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ON THE MOTION OF A LINE COMMON TO THREE DIFFERENT MATERIALS

by

Elizabeth B. Dussan V.

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland
1972

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TO BEN

Abstract

When the interface between two immiscible fluids meets a solid wall (e.g. air, water, and glass in a partially filled glass of water), a line common to all three materials is formed. This study examines, from a continuum point of view, the motion of the fluid in the neighborhood of a moving common line.

In several experiments performed in this study utilizing various fluids, it has been observed that fluid points initially on the fluid-fluid interface are transported to the common line in a finite amount of time. Such motion for viscous fluids would seem impossible for two reasons:

(i) The very idea of a moving common line seems to imply a violation of the no slip boundary condition.

(ii) Solid boundaries are usually thought of as fluid material surfaces. Proofs in the literature of such a statement are shown in this study to be either incorrect due to the use of an incomplete mathematical modelling of the definition of a material surface or incomplete due to high degree of smoothness assumed for the motion of the fluids.

Theoretical consequences of the above mentioned experimental observation are investigated. It can be shown that if the fluid in contact with the rigid boundary does not slip, then there must exist a multi-valued velocity field at the common line, and in fact, the normal component of the velocity

of the fluid at the common line need not be zero. All this is shown to be possible without violating the basic concepts of a continuum and of a bounding surface. These results are independent of any assumptions about the constitutive equation of the bulk fluids (e.g. Newtonian, non-Newtonian, viscoelastic) and independent of any jump conditions on the free interface (e.g. surface tension, surface viscosity, surface elasticity). Thus, the result is valid even for contaminated interfaces. In fact, there need not exist even a distinct mathematical interface. It is only sufficient that a fluid point in the interfacial region be transported onto the bounding wall in a finite amount of time and that there be no slip.

If, in addition, the fluids are assumed to be incompressible, then it is shown theoretically that there exists at least one surface (other than the fluid-fluid interface), which coincides with trajectories of fluid points originating from the common line and extending into one of the fluids. This surface has been observed experimentally in this study.

In order to be able to make quantitative predictions of the motion of the fluid (e.g. the velocity of the common line, the shape of the interface) it is necessary to make constitutive assumptions about the fluids and wall. It was decided to consider Newtonian incompressible fluids and a rigid bounding wall. However, it is proven in this study that such materials give use to an infinite force on the wall:

(i) For the full Navier-Stokes equation subject to certain weak restrictions.

(ii) For the Navier-Stokes equation with the acceleration terms neglected (Stokes flow) subject to no restrictions.

It is emphasized that these results are all independent of the jump conditions and the shape of the free interface (except that the contact angle must be between α_0 and π for case (i) $\alpha_0 = 0$, for case (ii) $60^\circ < \alpha_0 < 90^\circ$).

Acknowledgments

The influence of one's friends can never be underestimated. It is through open and free discussions that different "eyes" within one's own mind are opened enabling one to examine a particular concept from varying points of view. This, I feel, is an essential ingredient in research. It is for this reason I would like to thank:

Professor Stephen H. Davis, who has always been available for critical engaging discussions; who does not mind doing things which are a little different, a little unusual, and a little wacky; and who made my last three years at Johns Hopkins absolutely enjoyable.

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* Graduate Student in Mechanics Blues

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Introduction

When the surface formed between two immiscible fluids joins a solid boundary, a line is formed. This line is sometimes known as the three-phase line or the contact line; in this study it is referred to as the common line. The motion of the materials in the immediate vicinity of a moving common line is not well understood. One major cause of difficulty is that people in mechanics cannot reconcile having both the no slip boundary condition at the fluid-solid boundary with a motion that demands one fluid to displace another on the solid boundary. This is thought by some to be kinematically incompatible. In an investigation made by Ludviksson and Lightfoot (1971) of a thin film spreading up the sides of a vertical non-isothermal plate partially submerged in a pool of squalane, it is stated that due entirely to the no-slip condition between the liquid and solid wall, hydrodynamics cannot describe the flow in the neighborhood of the leading edge of the fluid. Therefore, the leading edge must advance by diffusional processes. However, in their analysis, they ignore the diffusional process because it is poorly understood and they use an ad hoc technique of extending their analysis up to the leading edge of the liquid film. They used a similar rationale when investigating the rise of a column of liquid in a capillary tube (1968). Prutow and Ostrach (1971) studied the movement of a fluid-fluid interface of two displacing fluids in a tube.

Their analysis is limited to receding interfaces having small contact angles. Their asymptotic analysis divides the flow field into an inner and an outer region. Their inner region consists of a three-phase region together with an adsorbed or deposited film on the walls of the tube. The thin film is modeled as a three-dimensional Newtonian fluid, i.e. it is a lubrication layer. In this way the whole problem of viscous fluids displacing each other at the wall is avoided. Schonhorn, Frisch and Kwei (1966) investigated experimentally the rate of spreading of polymer melts on aluminum and mica surfaces. A small drop of a polymer (polyethylene and ethylene-vinyl acetate) was placed on a horizontal surface. They found that in all cases as the drop spread, its configuration followed very closely that of a spherical cap. No thin film or "foot" extending from the drop on the surface which might arise from a "surface diffusional" type process could be observed.

Lord Kelvin (1848) investigated fluid at boundaries from strictly kinematic point of view. He derived necessary conditions for a surface to be a bounding surface of the fluid. He also emphasized the point that there is no reason to conclude that fluid material points once in contact with the surface, always remain in contact. Truesdell (1951) extended Kelvin's analysis and claimed to show that if the fluid is further restricted by assuming mass conservation, no sources or sinks of mass, and a density function that is never infinite or zero,

then the bounding surfaces are fluid material surfaces.

Huh and Scriven (1971) investigated the displacing of one fluid by another from a dynamic point of view. They assumed that the bounding surface is planar, the flow in the neighborhood of the common line can be described by two dimensional, steady, incompressible Stokes flow, the fluid-fluid interface is planar and the fluids obey the no slip condition at the bounding surface. Three basic difficulties arise in their analysis: (i) the velocity field of the fluid is not well-defined at the common line, i.e. multi-valued. (ii) the fluid exerts an unbounded force on the bounding surface in the neighborhood of the common line. (iii) the jump condition in the normal stress along the entire interface is not satisfied. Huh and Scriven are quick to point out the above difficulties and they attribute them to a failure of some of their basic hypotheses in their model. However, they cannot pinpoint where this failure occurs.

In Chapter I of this study, different physical situations, which possess moving common lines, are examined. A functional definition of the common line is given which is no more than a macroscopic, naked eye technique of locating its position. At the end of the chapter a qualitative experiment is described which strongly suggests that as the common line moves over the bounding surface, it does not necessarily leave behind a lubrication layer of the displaced fluid. This does not preclude the possible deposition of a "monomolecular layer" which, from

a continuum point of view could be modeled as a two-dimensional differentiable manifold.

In Chapter II a "Basic Assumption" is made which lays the foundation for the rest of this study. It is assumed that the common line is not a material line; fluid points on the fluid-fluid interface are constantly being mapped in a finite time onto the common line or vice versa. Results from four qualitative experiments are presented to support such a hypothesis. The experiments involve liquid-liquid and liquid-gas systems.

In the rest of the study the position is taken that the motion of the fluids in the neighborhood of a moving common line must possess the kinematical condition expressed in the Basic Assumption. Necessary characteristics of such a motion are investigated.

In Chapter III it is demonstrated that if the solid bounding wall is rigid and planar and if the fluid in contact with the solid is assumed not to slip, then the velocity field at the common line is not well-defined; specifically, it is multi-valued. This result is a direct consequence of the Basic Assumption and no slip; the multi-valueness is kinematically necessary for these two concepts to be compatible. There is no need to introduce concepts such as conservation of mass, linear momentum or interfacial conditions to derive this result.

In Chapter IV it is shown that for a surface to be a

bounding surface, it is not necessary that $\underline{u} \cdot \underline{n} = v_n$ at every point on the surface; in particular, this is not necessarily true at the common line. (v_n is the normal velocity of the surface, $\underline{u} \cdot \underline{n}$ is the normal component of fluid velocity at the wall.) As a direct consequence, it is seen that if $F(\underline{x}, t) = 0$ describes a bounding surface, it is not necessary that $\frac{dF}{dt} = 0$ everywhere on $F = 0$. A critical discussion is given concerning a fundamental definition of a bounding surface. It is shown that $\underline{u} \cdot \underline{n} \neq v_n$ at the common line does not violate this basic definition.

The latter part of Chapter IV is devoted to a careful review of Truesdell's (1951) results. It is shown that due to an incomplete definition of a material surface, the assumptions of conservation of mass and a continuous velocity field are not sufficient to insure that a surface $F = 0$ in which $\frac{dF}{dt} = 0$ is a material surface. An illustration is given which points out the incompleteness of the proof. In fact, it is shown that motions consistent with the Basic Assumption expose the weakness in the proof.

In Chapter V the fundamental theorem in non-linear ordinary differential equations, the Cauchy-Picard theorem, is applied to the moving common line problem. Due to the demand of non-uniqueness in the motion at the common line, it is a necessary consequence that the velocity field must not be Lipschitz continuous at the common line.

In Chapter VI it is demonstrated theoretically that a material surface must be emitted from the moving common line into one of the displacing fluids. This demonstration is based on the assumptions of no-slip, and mass conservation without sources, sinks or densities which could reach zero or infinity. It is emphasized that this result is independent of any assumptions concerning conservation of linear momentum and the jump conditions at the fluid-fluid interface, besides, of course, the Basic Assumption. Results of qualitative experiments are presented to demonstrate the plausibility of such a flow field.

In Chapter VII it is shown in two-dimensional incompressible flow that if the velocity field is multivalued at the common line and if the velocity field possesses a minimal amount of structure, then a certain integral is unbounded. For the case of a Newtonian fluid (and possibly for certain non-Newtonian fluids as well), this integral is the force exerted by the fluid on the wall. In other words, if one looks for a solution to the Navier-Stokes equation and requires the velocity field to have a certain representation, then, independent of jump conditions at the fluid-fluid interface (the interface can be "dirty", etc.), it is necessary that the velocity field gives rise to infinite forces.

If it could be proved that all solutions to the Navier-Stokes equation must possess velocity fields with the structure

described in Chapter VII, then one would know that the following statements are physically incompatible because they give rise to infinite forces; (i) the no slip condition on the rigid bounding surface, (ii) the Basic Assumption. However, this could not be done, and so a less general problem is considered. In Chapter VIII it is assumed that the fluid in some finite neighborhood of the common line is undergoing Stokes flow, i.e. a two-dimensional "slow" motion of two incompressible fluids. The resulting stream functions which represent the velocity field obeys the biharmonic equation. This can, in a sense, be considered a generalized form of the problem investigated by Huh and Scriven. However, there are fundamental differences in posing the problem. In this study it is assumed that the velocity field is multivalued at the common line. (We have seen that this is a direct consequence of the no slip condition and the Basic Assumption). It is not assumed that the fluid-fluid interface is flat. No assumptions need be made concerning the jump conditions at the interface. It is shown that if there exists a solution to the biharmonic equation in a finite domain containing parts of the two fluids and including the common line and if the velocity field, evaluated on the fluid-fluid interface, is Hölder continuous (except at the common line), then at least one of the fluids exerts an infinite force on the solid bounding surface. As Huh and Scriven suspected, it is the multi-valued velocity field at the common

line which gives rise to the infinite force. Any contrivance, such as a slip coefficient, which removes the multi-value velocity at the common line simultaneously removes the singularity in the force.

In order to establish the above mentioned result, the behavior of the solution to the biharmonic equation in only one of the displacing fluids is investigated.

There are two difficulties in the analysis; (i) the velocity field on the boundary of the fluids is discontinuous, and (ii) the domain has a corner. The first is overcome by decomposing the stream function in two parts. This reduces the problem to that of a continuous velocity on the boundary. The big difficulty is that the domain must contain a corner. The corner is located at the same point on the boundary of the domain at which the common line is located. The traditional techniques for analyzing the biharmonic equation are only applicable for domains with smooth contours. With the aid of Muskhelishvili's (1963; p.110) theorem on the representation of biharmonic functions, Magnaradze (1938b) derived an equivalent integral equation. This equation, due to the corner in the contour, does not possess a completely continuous operator. Consequently, a technique due to Radon (1919a) must be used. The integral operator is written as the sum of two operators, one of which is completely continuous; the other contains the singularity. The method of solution requires the norm of the

singular part to be less than one. In addition, in order that the solution to the integral equation give rise to a solution to the biharmonic equation it is necessary for it to be Hölder continuous. Magnaradze (1938b) seems to use an incomplete definition of the norm of the integral operator and so does not establish either the theorem nor that the solution is Hölder continuous. These parts of the analysis are supplied in Chapter VIII.

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I. Definition of the Common Line.

A day never goes by without witnessing a moving common line. Shortly upon awakening in the morning it is customary to partake in a western ritual...ingestion of a cup of coffee. As the coffee is being poured we can distinguish a moving line common to the coffee, air, and cup. Perhaps this feast includes a piece of toast with jam. The utensil used to spread the jam hopefully was initially clean, that is, in contact with the air. In the process of loading it up with jam, the air is displaced. This illustrates another case of a moving common line.

It is through personal experience that we are aware of the physical phenomenon of one material displacing another at a solid bounding surface. In the first example it was the coffee displacing the air on the inside surface of the cup. In the second example it was the jam displacing the air on the surface of the utensil. The mere washing of the dishes at the end of breakfast attests to the fact that indeed the coffee had come in physical contact with the cup and, hence, must have actually displaced the air.

The most familiar cases of two mutually displacing materials at a solid boundary involve air and some kind of liquid. Turn on a faucet and you see water displacing air on the surface of the sink. Drop a container of milk and most likely the air is displaced by the milk on the surface of the

floor and you have one hell of a mess to clean up. Wake up early on a summer's morn, and watch a drop of dew making its way down a blade of grass.

There are some less obvious examples in which both displacing materials are two distinct liquids. Pour into a glass container first some vinegar, then some oil. One can observe, as the container is gradually tilted from its original position, the oil being displaced by the vinegar. The above experiment works just as well if the vinegar is replaced by either water or alcohol (99% pure isopropyl; methyl or 100 proof Smirnoff Vodka).

The above mentioned demonstrations can be divided into two categories. The first consists of those systems of materials (one must specify three materials for each system) for which the common line can move in various directions along the solid surface, the specific direction being partially determined by the three-dimensional configuration of the displacing materials. A drop of water is placed on a horizontal surface of bee's wax. Upon tilting the surface the drop starts to move down hill. The advancing portion of the water is observed to displace the air while the receding end is displaced by the air. Consider again the oil on water in a glass or plexiglass container. Jostle the container a bit and the common line moves up and down the inside wall.

The second category consists of sets of materials for

which one material is always displacing the other, but never the reverse. A drop of 99% pure methyl alcohol is applied to a piece of glass or plexiglass completely submerged in oil. The drop tends to spread into a very thin film. The common line always moves in that direction which results in the alcohol displacing the oil. (However, if one waits long enough, the thin film of alcohol disappears owing to the partial solubility of the alcohol in the oil). This observation should be of no surprise to individuals who use alcohol as a cleansing agent. Some bee's honey is poured on a plexiglass surface. After a short while it appears as though the common line is not moving. If the surface is tilted, the front of the honey advances downhill displacing the air. On the other hand the uphill portion of the common line does not budge, that is, the air does not displace the honey.

At this point it is worth establishing in some qualitative sense a procedure for locating the common line. This will be considered as the functional definition of the common line. From a macroscopic, naked eye, point of view the common line lies at the intersection of the interface which divides the two mutually displacing materials with that of the solid bounding wall. The interface between the two fluids and the surface of the solid can be identified and differentiated from each other by the dissimilar manner in which they reflect light. (The two displacing materials shall henceforth be referred to

as fluids, in the layman's sense of the word. This will not imply any restriction on the form of their constitutive equations which relate stress with other variables. In the same spirit, the material which makes up the boundary will be referred to as a solid.) Let us consider the case of silicone oil* and water in a plexiglass container as illustrated in figure (2.8). The fluids occupy a rectangular container which is inclined at an angle α_0 with respect to the horizontal. (Unless otherwise specified, the horizontal always refers to a surface perpendicular to the direction of gravity.) It is observed that the character of the light reflected off that part of the plexiglass surface which has always been in contact with the water is the same as that off the portion of plexiglass originally in contact with the oil but currently in contact with the water, i.e. the portion of plexiglass in which the common line has moved over. By quickly forcing the bottom of the tank to an almost horizontal position the oil-water interface bulges into the oil as a consequence of their different densities. However, the common line is comparatively slow in responding to this change in configuration, as shown in figure (1.1). Even though only a thin layer of oil is located underneath part of the water, there is no confusion as to the location of the common line.

* Union Carbide L-45 Silicone fluid, $\nu = 100$ ctsks.,
 $\rho = .97$ grams/cm³

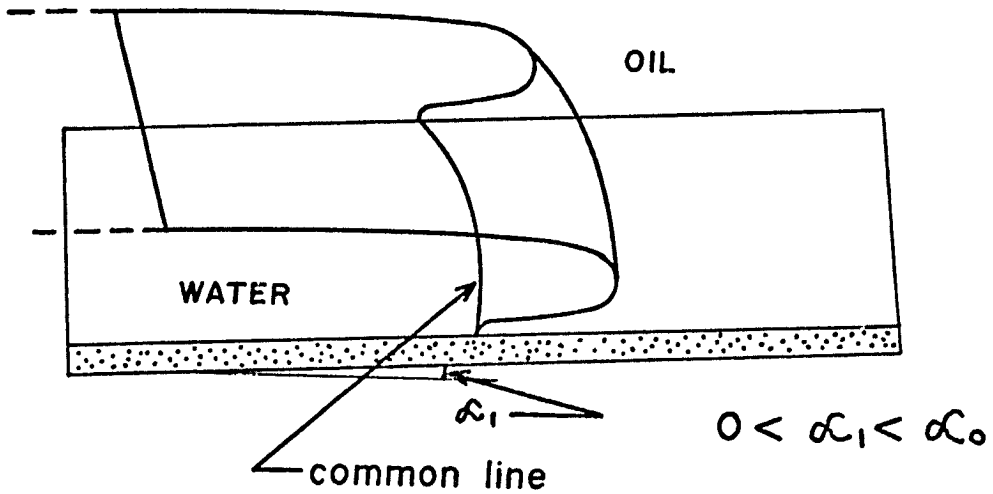


Fig. (1.1) The fluid-fluid interface bulges into the oil immediately after forcing the tank to an almost horizontal position.

The above demonstration should not be construed as an attempt to establish beyond a shadow of a doubt that no oil molecules exist on the entire water-solid interface. This includes that portion of the solid over which the common line has passed. Consider the case of a capillary tube partially immersed in water. A column of water rises to some equilibrium height. It is popular when analyzing its static equilibrium configuration to consider the water in contact with the inside wall of the capillary, even though gas molecules probably can be found there. This is the type of difficulty that arises when modeling materials. We must first formulate the questions

we want to answer; only then can we attempt to model the system. In the case of the capillary tube, if we want to know the height of the column of water or the slope of the water-air interface, then it is appropriate to model the air and water as continua with the water in contact with the capillary, and the walls of the capillary as a rigid solid. For the answers to the above questions, a molecular model of the materials would probably not be too useful.

Ultimately we will like to be able to understand the mechanism which governs the motion of the common line (the word motion has a specific meaning which is discussed in the next section). In this study the point of view is taken that it is appropriate to model the materials involved as three-dimensional continua.

If we regard the above mentioned procedure of locating the common line as its functional definition, then an important point to corroborate in the oil-water system is that there does not exist a lubrication layer of oil on the portion of the surface in which the common line has passed over. (The expression "a lubrication layer of oil" means a layer of oil which must be modeled as a three-dimensional continuum and whose deformation can be predicted by the Navier-Stokes equation.) For this purpose let us consider another qualitative experiment involving oil-water and plexiglass but in a slightly different configuration than above. A drop of water approxi-

mately .1 cc in volume is released in the middle of a tank completely filled with silicone oil. The drop of water contains food dye for photographic purposes; refer to figure (1.2).

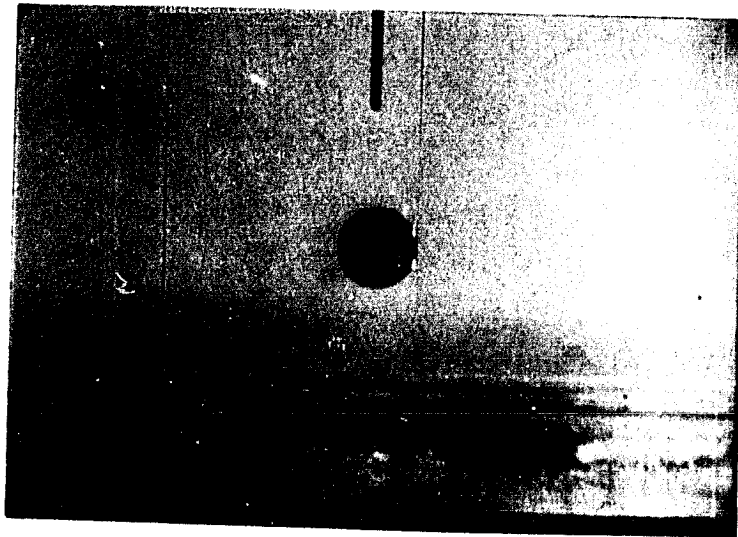


Fig. (1.2) A drop of water containing food dye is released from a hypodermic needle. The drop is surrounded by silicone oil ($\nu = 10$ ctsks). The bottom surface is plexiglass.

The drop remains relatively spherical as it falls. For 5 to 10 seconds the drop appears, to the naked eye, to be resting on the bottom of the tank. All of a sudden the bubble of water "pops" on the plexiglass surface and achieves an entirely different configuration; refer to figure (1.3). During the interval of time in which the drop first appears to be in contact with the lower surface it most likely is not. Chances

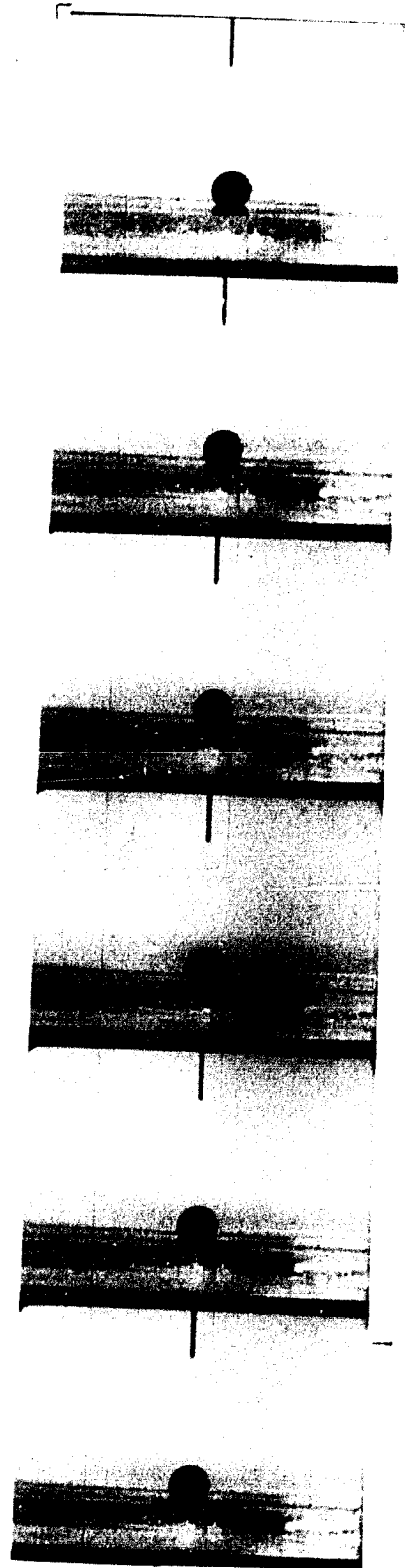
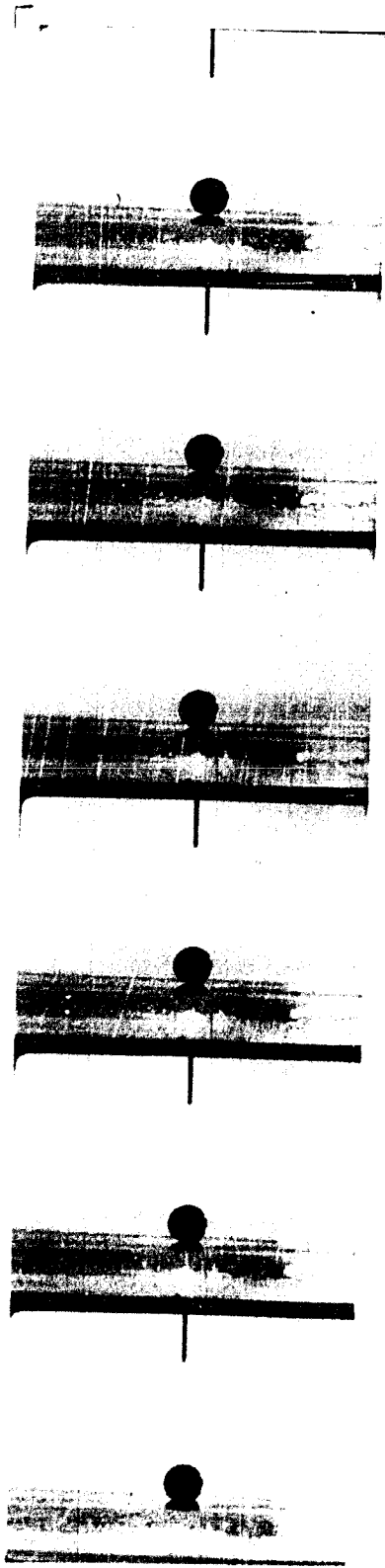


Fig. (1.3) The drop "popped" in less than .75 sec.
The motion picture was taken at 18 frames per sec.

are that a very thin lubrication layer, imperceptible to the naked eye, exists between the drop and the surface. After this layer of oil has reached a critical thickness, something occurs resulting in a radical change in the configuration of the drop. It seems reasonable to conclude from this demonstration that it seems inappropriate to mathematically model the process of "popping" by the Navier-Stokes equations with classical boundary conditions in which it is implicitly kinematically assumed that a thin layer of oil always remains between the water and the bottom surface. Hence, it might be reasonable, depending on the types of questions one is trying to answer, to model the "popped" drop by water directly in contact with plexiglass.

Let us now investigate in more detail the deformation of the fluids when the drop of water is in the process of "popping". To slow down the motion, the drop consists of glycerine (about 1500 times more viscous than water); refer to figure (1.4). Compare the appearance of the light reflecting off the lower surface of the water drop in the initial picture with that in the final picture. Notice the way the line dividing these two distinct types of reflections sweeps across the lower surface of the drop during the popping

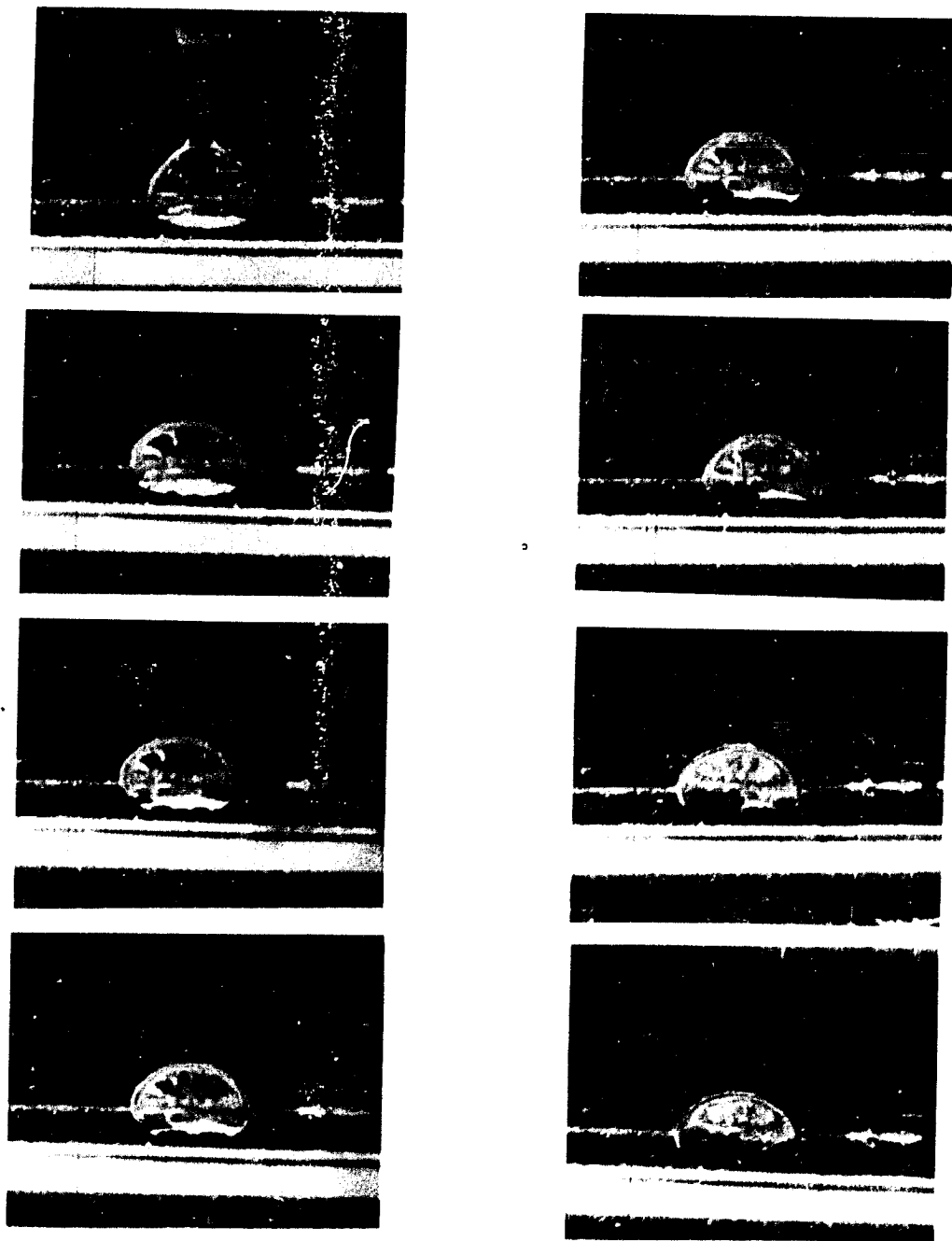


Fig. (1.4) Glycerine drop "popping" on a plexiglass surface surrounded by silicone oil.

process. In fact, this looks just like the first demonstration of the moving common line (refer to figure 1.1). Hence, the fact that it is quite unlikely that the popped drop possesses a lubrication layer of oil under it strongly suggests that it is equally unlikely that there is any lubrication layer of oil between the water and the plexiglass in the first experiment.

II. The Basic Assumption in the Theory.

All materials in this study are modeled as continua. This means that the "real" stuff, the atoms and molecules, are mathematically replaced by infinitely divisible media. To describe the motion of a continuum we use a function,

$\underline{x} = \underline{\chi}(\underline{R}, t)$ which maps the position of a material point (not to be confused with a molecule) at some past time into its position at time t . For definiteness it is assumed that $\underline{R} = \underline{\chi}(\underline{R}, 0)$ and so $\underline{\chi}(\underline{R}_0, t)$ is the position at time t of a specific material point which was located at \underline{R}_0 at time $t=0$. The curve $\underline{x} = \underline{x}(t) \equiv \underline{\chi}(\underline{R}_0, t)$ is called the trajectory of this material point. This is illustrated in figure (2.1).

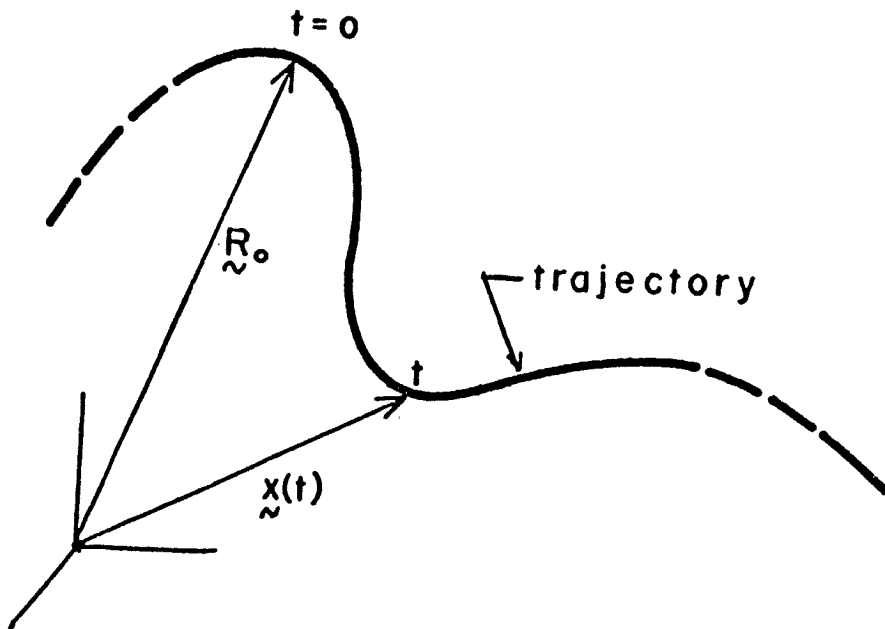


Fig. (2.1) The trajectory of a material point located at \underline{R}_0 at $t=0$. The vector $\underline{\chi}(t)$ is its position vector, and $\underline{R}_0 = \underline{\chi}(0) = \underline{\chi}(\underline{R}_0, 0)$.

It will be assumed that this trajectory varies continuously with time; that is, the function $\mathcal{V}(\underline{R}, t)$ is continuous with respect to t for any fixed value of \underline{R} .

Up to now the three materials involved in the moving common line phenomena have been referred to as "distinctly different". Each fluid possesses its own set of physical constants such as density, viscosity, thermal conductivity, etc. When the two fluids make contact with each other, it is assumed that an interface is formed. It is customary to model this situation by two distinctly different, three-dimensional fluids separated by some two-dimensional interface. The scalar-valued functions describing the physical constants are then single-valued and well defined within the bulk of the fluids; however, they become multi-valued on the interface. This is the manner in which almost all fluid mechanics have envisioned a system with two immiscible fluids. Yet, there have been some attempts to model the interface as a three-dimensional region (Maxwell, 1876; Scriven, 1971) of exceedingly small thickness where the physical constants undergo a very rapid, but nevertheless continuous changes from one fluid to the other. (A similar comment can be made about the interface formed between the solid wall and the surrounding fluids). Be that as it may, there are some features of the motions of the fluids in the neighborhood of the common line whose description is independent of which model one chooses to use for the inter-

facial region. The first of these is:

Basic Assumption: Material points on the fluid-fluid interface (or, in the interfacial region) are mapped onto the common line (or common line region), or vice versa, in a finite interval of time.

In order to understand this assumption better, let us express it in terms of the motion of the fluid points for the case when the interfaces are modeled as two-dimensional surfaces. Define $\gamma_{c.l.} \equiv \underline{f}(s, t)$ to be the location in space at time t of the common line (S is an arbitrary parameter). Consider a material point located on the common line at time $t=0$, specifically, that material point located at $\underline{f}(s_0, 0)$. It is important to realize that this material point has three identities. This is a direct consequence of the fact that a material body is considered to be the closure of an open set. Hence, the material points on the fluid-fluid interface have a double identity, while the material points which lie on the common line have a triple identity.

The trajectory of the material point located at position

$$\underline{f}(s_0, 0) \text{ at time } t=0 \text{ is } \gamma_{c.l.} = \gamma_{c.l.}(\underline{f}(s_0, 0), t).$$

The "Basic Assumption" states that for a moving common line, there exists a time t_0 , $|t_0| < \infty$, such that

$\gamma_{c.l.}(\underline{f}(s_0, 0), t_0)$ is located on the fluid-fluid interface and not on the common line. If $t_0 < 0$, then the material point is said to be mapped from the interface onto the

common line; if $t_0 > 0$, then the material point is mapped from the common line onto the interface (this is the "vice versa" part of the assumption).

We shall now describe four experiments which demonstrate qualitatively that the above is probably a characteristic of at least a class of systems which possess a moving common line.

About 1 cc. of honey is placed on a horizontal plexiglass surface. In this system the two displacing fluids are air and honey, and the solid bounding wall is plexiglass.

A small dye mark, which consists of a honey and food dye mixture (McCormick's) is placed by means of a sewing needle on the air-honey interface at the plane of symmetry of the honey; see figure (2.2).

The dye mark appears initially to be circular and slowly tends to increase in size possibly due to a surface tension gradient created by the presence of the food dye. The plexiglass is tilted and the honey starts moving downward (see figure (2.2)). The trajectory of the dye mark is photographed from the direction indicated in figure (2.2). The camera is inclined at a slight angle pointing down to the drop. As a consequence of the downward tilt of the camera, we can see a mirror image of the drop on the plexiglass surface. The common line can be located by eying in the line formed by the intersection of the honey-air interface with its reflected image; see the first picture in figure (2.3). As the honey moves

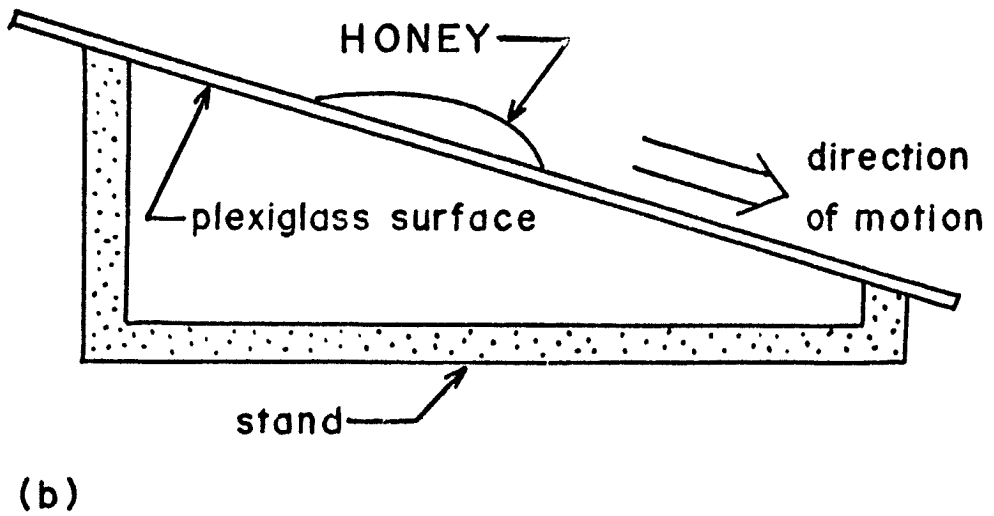
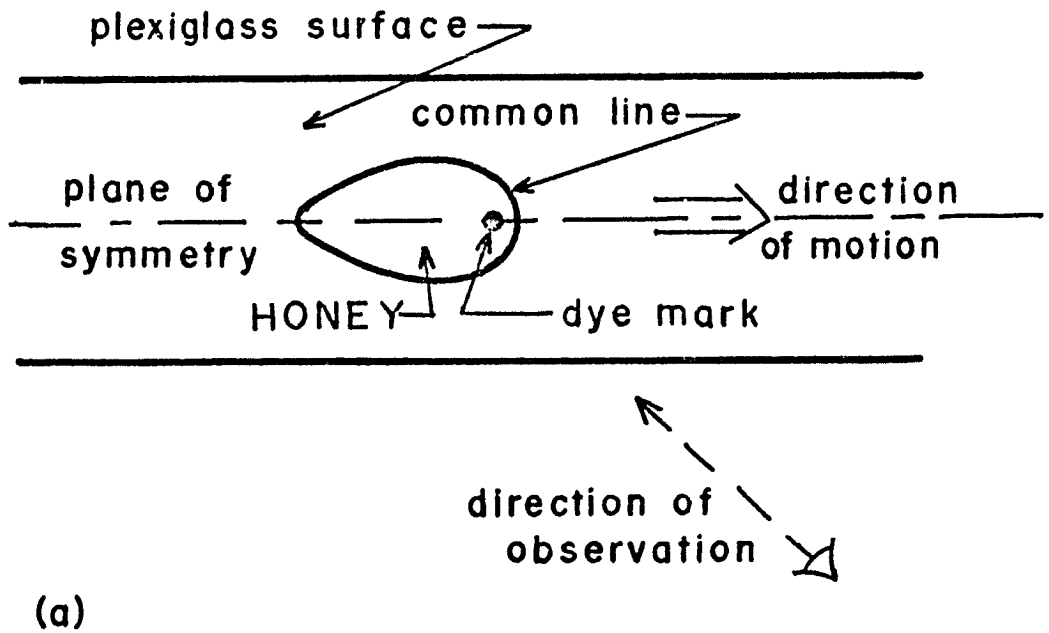


Fig. (2.2) (a) Plan view and (b) side view of a drop of honey on a plexiglass surface.



Fig. (2.3) A drop of honey flowing on a plexiglass surface.

forward, so does the dye mark. Throughout, the dye mark roughly retains its circular geometry. Finally, in picture 3 of figure (2.3) the dye mark makes contact with the plexiglass, and therefore constitutes part of the common line. In picture 4, more than half of the dye is in contact with the plexiglass surface and the remaining part is still on the honey-air interface; the reflected image of this latter portion makes the mark appear to be a small dark spot. The last picture shows no dye mark at all. It is important to note that the dye mark does not spread out in the direction parallel to the plexiglass surface as it approaches the common line; we shall come back to this point later. We have thus observed material points on the fluid-fluid (honey-air) interface, i.e., those points of material (honey) marked by the dye and which also lie on the interface (such points probably exist since there is a tendency for the dye mark to grow in size) to follow a trajectory which brings them to the common line in a finite length of time.

In the first experiment, the moving common line involved a liquid (honey) displacing a gas (air). A case in which a gas (air) displaces a liquid (glycerine) is now examined; this time the solid surface is made of bee's wax. The surface is prepared by pouring molten wax into a dish and letting it solidify and cool to room temperature. It should be noted that on close inspection, by reflecting light off the surface at different

angles, a definite crystalline structure can be observed. Two drops of glycerine, one transparent and the other dyed (the same food dye as in the previous demonstration) are placed side by side on the wax surface; see figure (2.4).

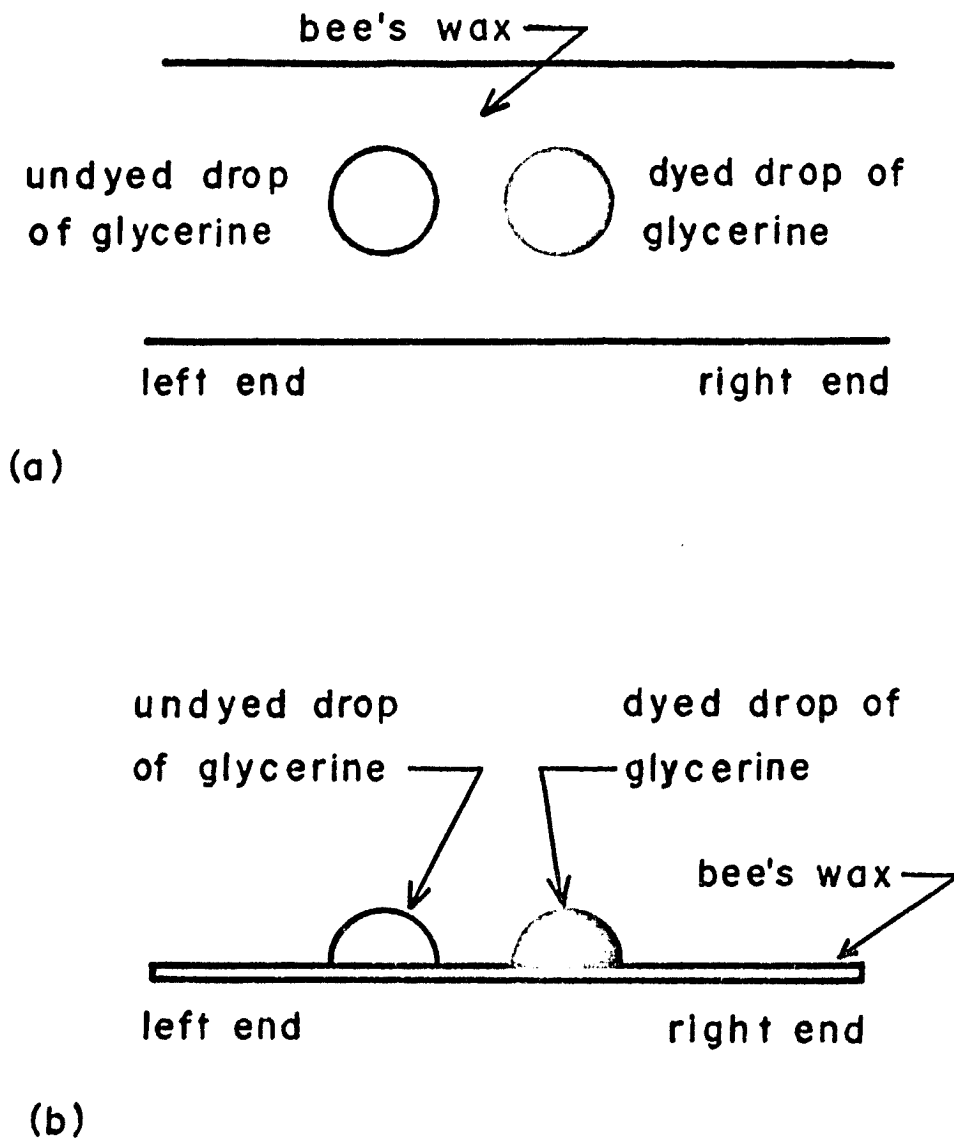


Fig. (2.4) (a) Plan view and (b) side view of two drops of glycerine. One drop is dyed, the other is transparent.

The wax surface is tilted from the horizontal causing the right end to be lower than the left end (see figure (2.4) for definition of right and left ends). After a very short time, the two drops merge, whereupon the solid surface is made horizontal again; see pictures 1 and 2 in figure (2.5).

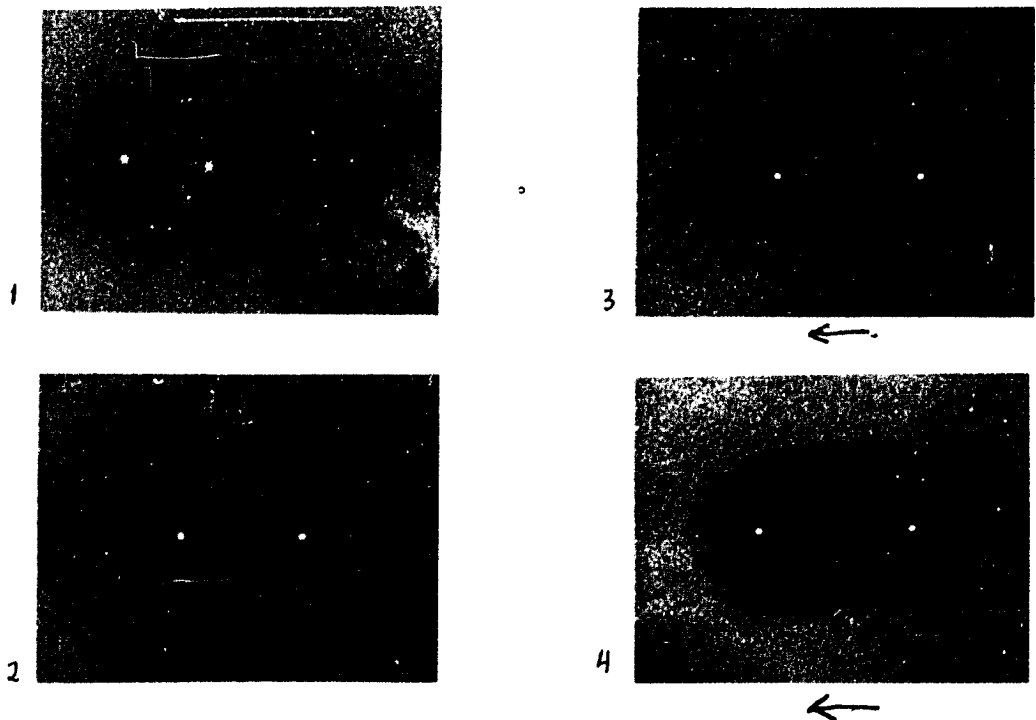


Fig. (2.5) (1) Two drops of glycerine on a bee's wax surface. (2) Now there is one drop, part of which is dyed. The arrows beneath (3) and (4) indicate the direction of movement of the drop of glycerine.

A cross-sectional side view would probably look like figure (2.6). Only one drop of glycerine is now present; it has approximately half of its mass marked by dye while the other half remains transparent. The dye tends to diffuse slowly into the

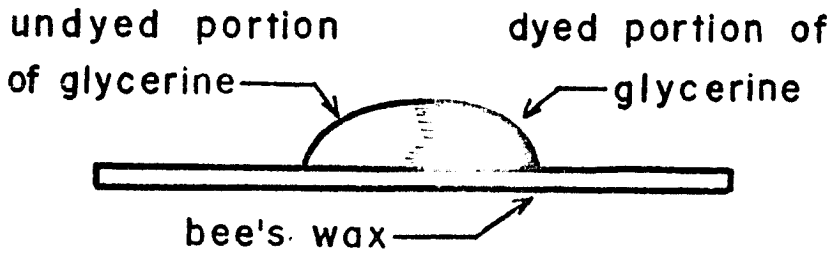


Fig. (2.6) Cross sectional view of a drop of glycerine which is partially dyed.

clear portion. The important point to keep in mind is that part of the common line of the drop (the right side) is composed of dyed glycerine, while the left side is composed of clear glycerine. In fact, the glycerine, just in contact with the wax surface, probably looks like figure (2.7).

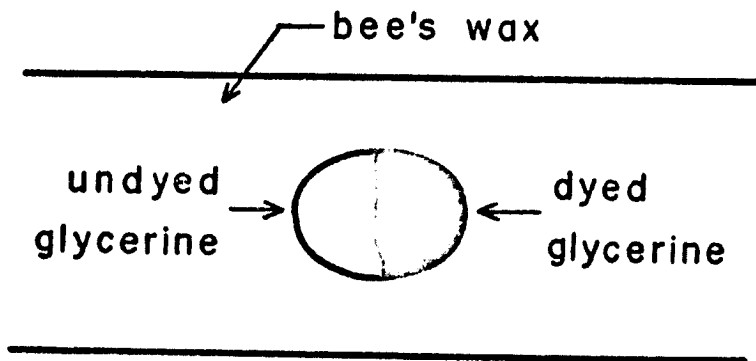


Fig. (2.7) Possible appearance of glycerine drop as viewed from below, as if the wax were transparent.

The right end portion of the glycerine in contact with the wax is dyed while the left end portion is clear. The experiment

proceeds by lowering the left end of the wax so that the glycerine moves to the left; see picture 3 in figure (2.5). After a finite length of time it is observed that the entire common line is composed of clear glycerine; see picture 4 in figure (2.5). This must indicate that the dye material points which were located on the right end portion of the common line must have moved someplace else; on careful observation it is found that this material is mapped onto the glycerine-air interface.

It is not necessary for one of the fluids to be a liquid and the other a gas. The behavior illustrated in the first demonstration, that with honey on plexiglass, is observed for the system of glycerine, silicone oil and plexiglass. A rectangular container made of plexiglass is placed at an angle with respect to the horizontal; see figure (2.8). The container is first partially filled with glycerine and then with silicone oil. The photographs in figure (2.9) are taken from the direction indicated in figure (2.8) and at an angle looking down onto the bottom surface. A small amount of dye is placed on the glycerine-oil interface near the common line; see picture 1 in figure (2.9). The dye is composed of glycerine and food dye (McCormick's), the density of which is slightly less than the glycerine alone. This density variation will be evident later, in the last demonstration of this section. The right end of the container is slowly lowered. It is observed that the

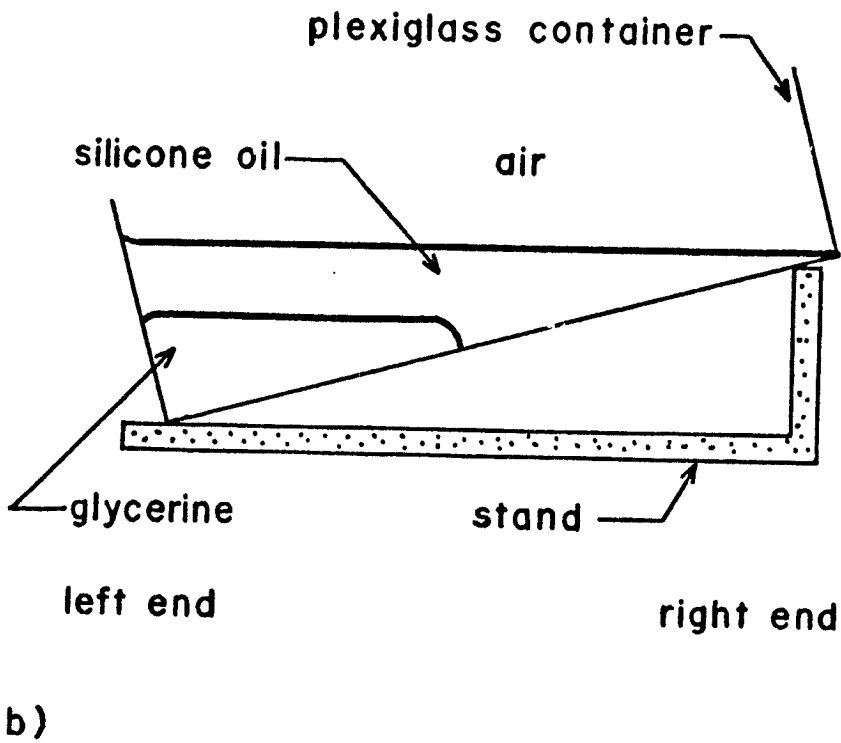
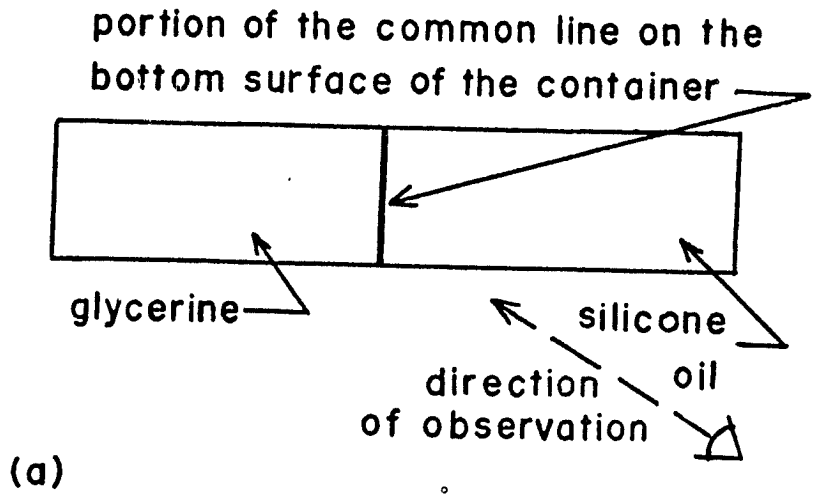


Fig. (2.8) (a) Plan view of the bottom surface of the container. (b) Side view of the container of glycerine and silicone oil.

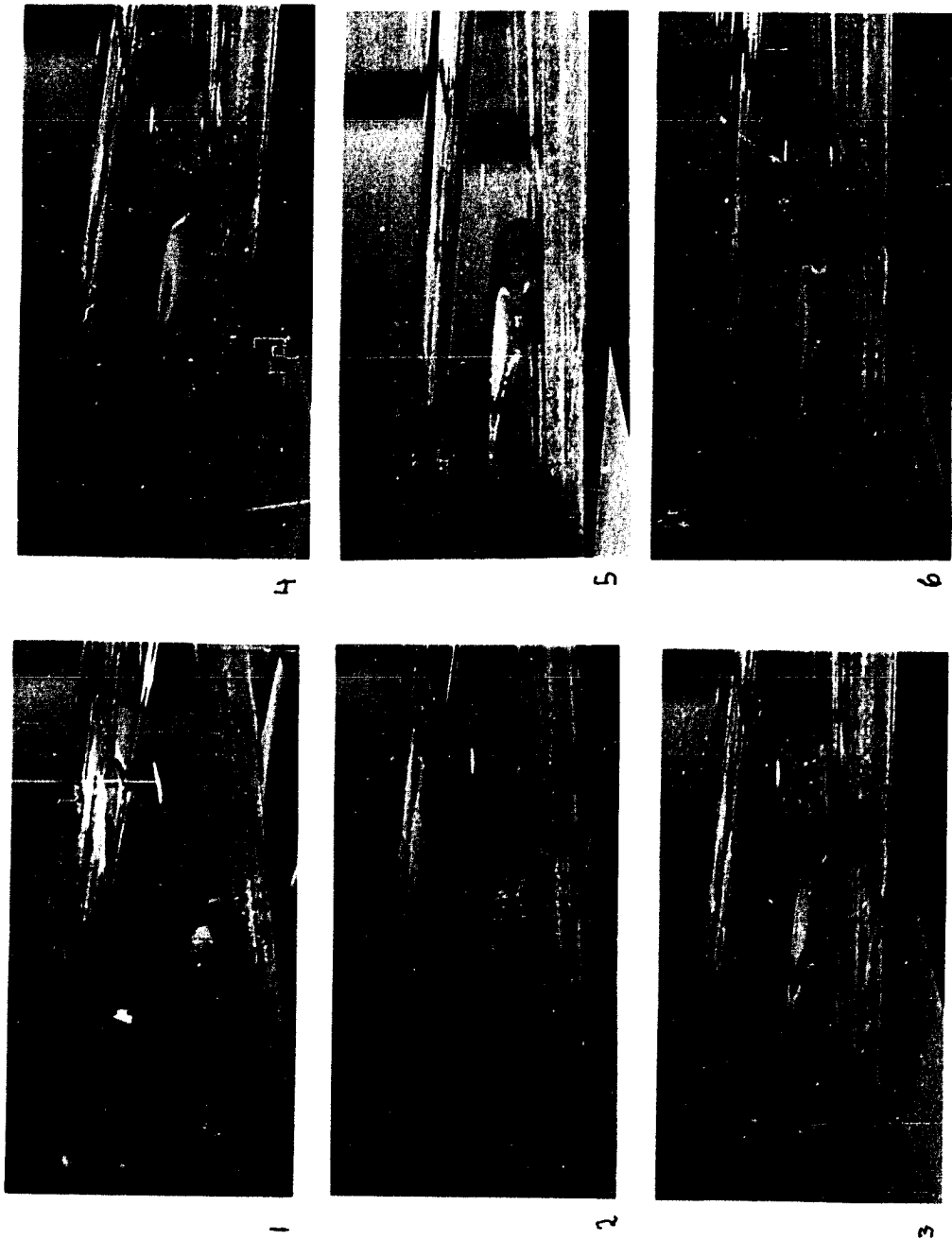


Fig. (2.9) The motion of a dyed piece of glycerine on the glycerine-oil interface. The glycerine is the lower fluid, and the oil is the upper fluid.

common line moves forward (to the right) and the dye mark tends to approach the common line; see pictures 2 and 3 in figure (2.9).

As time progresses the dye mark becomes part of the common line (picture 4 in figure (2.9)) and finally disappears from sight; it is no longer on the glycerine-oil interface. Two things should be noted: (i) The shape of the mark remains roughly circular as in the first demonstration of this section. (ii) The glycerine just beneath the interface (this includes dye at some positions) moves forward (to the right) slower than the material points on the interface. This is evidenced by the thinning of the dye mark at its forward position (see picture 3 in figure (2.9)).

The fourth and last demonstration involves the same materials as above; however, the common line is made to move backwards (to the left) by slowly raising the right end of the container. First a drop of dyed glycerine is deposited on the lower surface of the container which is initially covered with silicone oil. After a few moments, the drop "pops"; for a side view of the fluids, see picture 1 in figure (2.10). The right end of the container is lowered and the clear glycerine moves forward, eventually merging with the dyed glycerine (see picture 2 in figure (2.10)). Now the right end of the container is gradually raised (not lowered). We see, as a consequence of the dyed glycerine being less dense than the clear part, that most of the dye flows up the glycerine-oil interface. Despite

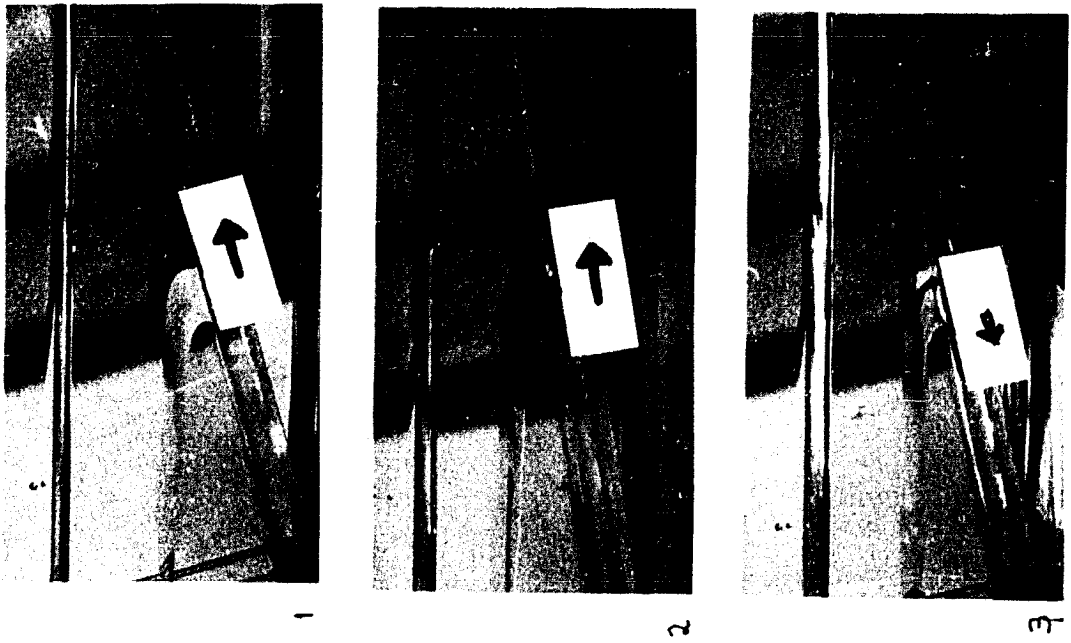
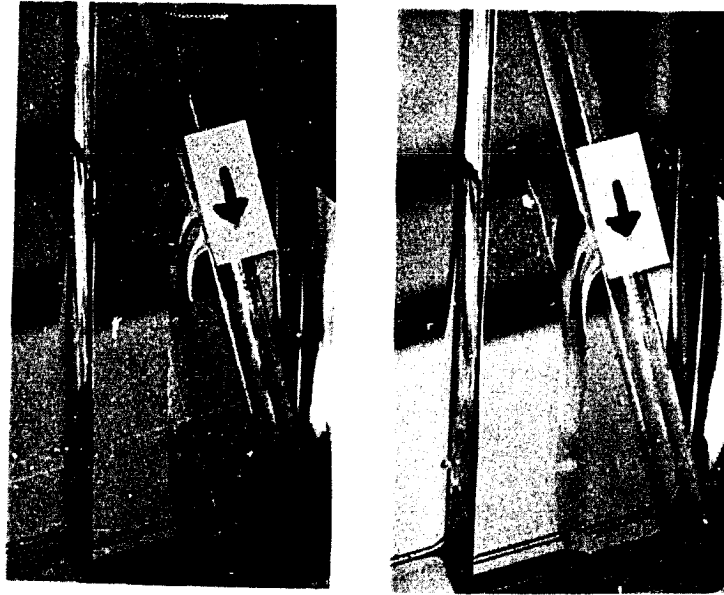


Fig. (2.10) The lower fluid and the dyed fluid are composed of glycerine. The upper fluid is oil. The arrows indicate the direction of motion of the common line.

this occurrence, the portion of the dye initially in contact with the plexiglass remains there; see picture 3 in figure (2.10). As the common line moves to the left, this remaining dye comes off the bottom surface. Finally, the entire mark is lifted off the bottom surface; see the last picture in figure (2.10). This illustrates again that material on the common line at one instant in time (the dyed glycerine) moves onto the fluid-fluid interface after the elapse of a finite interval of time.

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III. To show that the velocity field is not well-defined at the common line.

It has been established, through the qualitative experiments described in Chapter II, that the common line is not a material line (nor a material region), i.e. different fluid points are identified with the common line at different times. This is the foundation of the Basic Assumption in Chapter II. The next step is to examine how such a statement concerning the trajectory of the material points on the fluid-fluid interface and common line might affect or restrict the velocity fields of these materials.

Before proceeding, a qualification is placed on the Basic Assumption for the case when the interface is modeled as a three-dimensional region. It shall be assumed from now on that there exists a two-dimensional differentiable manifold of fluid points in the interfacial region which behaves just like the fluid points on the fluid-fluid interface in the two-dimensional case, i.e. they are mapped onto the common line in a finite interval of time (or vice versa). Hence the illustration in Chapter II of the Basic Assumption in terms of the motion of the material points for the case when the fluid-fluid interface is modeled as a two-dimensional region also applies to the three-dimensional model.

We shall start off by looking at systems in which the material composing the solid bounding wall is assumed to be

rigid and its surface to be planar. (Once the assumption of a rigid wall is made there is little loss in generality, in the following analysis, in assuming it to be planar). In addition, it is assumed that the two fluids do not slip at the wall, i.e. while the fluid points are in contact with the wall they undergo the same motion as the solid material points with which they initially come in contact. The fluid and solid material points at the interface occupy the same geometric points in space, that is, every point on the interface is identified as both fluid and solid. Nevertheless, this does not imply that their trajectories are the same for all time. The trajectory of the fluid point located at \underline{R}_s (\underline{R}_s is on the solid-fluid interface) at $t=0$ is $\underline{x}_f = \underline{x}_f(t)$ where

$$\underline{x}_f(t) \equiv \lim_{\substack{\underline{R} \rightarrow \underline{R}_s \\ \underline{R} \in \text{FLUID}}} \underline{x}(\underline{R}, t)$$

while the trajectory of

the solid point located at \underline{R}_s at $t=0$ is $\underline{x}_s = \underline{x}_s(t)$

where

$$\underline{x}_s(t) \equiv \lim_{\substack{\underline{R} \rightarrow \underline{R}_s \\ \underline{R} \in \text{SOLID}}} \underline{x}(\underline{R}, t).$$

(For the special case when \underline{R}_s is located on the common line at $t=0$ the point has three identities and there can possibly exist three (or more) separate trajectories originating from this point, i.e.

$$v_{\tilde{f}_1} = v_{\tilde{f}_1}(t) \equiv \lim_{\substack{R \rightarrow R_s \\ R \in \text{FLUID \#1}}} v_{\tilde{f}_1}(R, t) \quad ,$$

$$v_{\tilde{f}_2} = v_{\tilde{f}_2}(t) \equiv \lim_{\substack{R \rightarrow R_s \\ R \in \text{FLUID \#2}}} v_{\tilde{f}_2}(R, t) \quad ,$$

$$v_{\tilde{s}} = v_{\tilde{s}}(t) \equiv \lim_{\substack{R \rightarrow R_s \\ R \in \text{SOLID}}} v_{\tilde{s}}(R, t) \quad . \quad)$$

Thus the no slip assumption states that $v_{\tilde{s}}(t) = v_{\tilde{f}_1}(t)$ for all the times that the fluid point is in contact with the surface.

Let us now proceed to investigate the velocity of a fluid point whose trajectory lies on the fluid-fluid interface (or on the above mentioned surface imbedded in the interfacial region) and which at time t_0 lies on the common line. It is assumed that the fluid-fluid interface possesses a well-defined tangent plane at every point and that the common line has a continuous, well-defined tangent. A rectangular cartesian coordinate system is constructed such that its origin always lies on the common line, with the requirement that at time $t=t_0$ it coincides with the above mentioned fluid point. Otherwise, the trajectory of the origin is arbitrary but smooth. The coordinate system is oriented with its Y -axis in the direction perpendicular to the planar solid surface, its

Z -axis on the solid surface and parallel to the tangent of the common line at the origin, and the X -axis on the solid surface perpendicular to the Z -axis. In this coordinate system the fluid-fluid interface is represented by

$$K(X, Y, Z, t) = 0.$$

Let $X = X(t)$, $Y = Y(t)$ and $Z = Z(t)$ describe the trajectory of the fluid point. The requirement that it always lies on the fluid-fluid interface implies that

$$K(X(t), Y(t), Z(t), t) \equiv 0 \quad \text{FOR } t \leq t_0 .$$

If the above equation is differentiated with respect to time, then

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial X} \frac{dX}{dt} + \frac{\partial K}{\partial Y} \frac{dY}{dt} + \frac{\partial K}{\partial Z} \frac{dZ}{dt} = 0 \quad (3.1)$$

$$\text{OR, } v(t) = -\left(\frac{\partial K}{\partial Y}\right)^{-1} \left\{ \frac{\partial K}{\partial X} u + \frac{\partial K}{\partial Z} w + \frac{\partial K}{\partial t} \right\}$$

where u , v and w are the corresponding X , Y , and Z components of the velocity of the fluid point. At time $t = t_0$ the fluid point is located on the common line. As a consequence of the definition of the trajectory of the origin of the coordinate system we have

$$0 = X(t_0)$$

$$0 = Y(t_0)$$

$$0 = Z(t_0)$$

For the moment let us assume the one sided derivatives

$$\left. \frac{\partial K}{\partial Y} \right|_{\substack{\text{COMMON} \\ \text{LINE}}} \quad \text{and} \quad \left. \frac{\partial K}{\partial X} \right|_{\substack{\text{COMMON} \\ \text{LINE}}} \quad \text{do not vanish;}$$

we shall return to these cases later.

It follows from the definition of the coordinate system that

$$K(0,0,0,t) = 0$$

hence

$$\left. \frac{\partial K}{\partial t} \right|_{(0,0,0,t)} = 0, \quad (3.2)$$

where the order of the limit and the differentiation have been changed.

Since the Z -axis is chosen parallel to the tangent of the common line at the origin, it follows that

$$\left. \frac{\partial K}{\partial Z} \right|_{(0,0,0,t)} = 0. \quad (3.3)$$

Again, the order of the limit and the differentiation have been changed.

The quantities $v(t_0^-)$ and $u(t_0^-)$ are defined as follows:

$$v(t_0^-) \equiv \lim_{\substack{t \rightarrow t_0 \\ t < t_0}} v(t)$$

and

$$u(t_0^-) \equiv \lim_{\substack{t \rightarrow t_0 \\ t < t_0}} u(t)$$

Taking the limit of eqn (3.1) as $t \rightarrow t_0$ for $t < t_0$ and using eqns (3.2) and (3.3), it follows:

$$v(t_0^-) = - \left(\frac{\partial k}{\partial y} \Big|_{(0,0,0,t_0)} \right)^{-1} \frac{\partial k}{\partial x} \Big|_{(0,0,0,t_0)} u(t_0^-) \quad (3.4)$$

It is further assumed that once the fluid point makes contact with the solid surface it remains in contact for all subsequent times. Combining this with the no slip assumption, we have that the trajectory of the fluid point for $t > t_0$ is:

$$\begin{aligned} X &= X(t) = \int_{t_0}^t U(\tau) d\tau, \\ Y &= Y(t) = 0, \\ Z &= Z(t) = \int_{t_0}^t W(\tau) d\tau, \end{aligned}$$

where $U(t)$ and $W(t)$ are the X and Z components of the velocity of the solid with respect to the origin. $u(t_0^+)$ and $v(t_0^+)$ are defined as follows:

$$u(t_0^+) \equiv \lim_{\substack{t \rightarrow t_0 \\ t > t_0}} u(t) ,$$

$$v(t_0^+) \equiv \lim_{\substack{t \rightarrow t_0 \\ t > t_0}} v(t) .$$

It follows from above that

$$u(t_0^+) = U(t_0)$$

$$v(t_0^+) = 0$$

Let us compare the terms $u(t_0^-)$, $v(t_0^-)$, $u(t_0^+)$, and $v(t_0^+)$. If it is demanded that $v(t_0^-) = 0$, which means that the fluid point approaches the solid surface with zero normal velocity, then from eqn (3.4) it follows that $u(t_0^-) = 0$. But

$u(t_0^+) = U(t_0)$ where $U(t_0) \neq 0$ by virtue of the fact that the common line is moving. Hence we have a situation in which $u(t_0^-) \neq u(t_0^+)$.

Suppose, instead, that it is possible to have

$u(t^-) = u(t^+)$ (It is easy to see from the geometry of the interface that this is not always possible.)

Then it follows that

$$v(t_0^-) = \left(\left. \frac{\partial \kappa}{\partial Y} \right|_{(0,0,0,t_0)} \right)^{-1} \left. \frac{\partial \kappa}{\partial X} \right|_{(0,0,0,t_0)} U(t_0)$$

which means $v(t_0^-) \neq 0$ and hence $v(t_0^-) \neq v(t_0^+)$

Hence, it has been shown that $v(t_0^-) = v(t_0^+)$ and

$u(t_0^-) = u(t_0^+)$ can never be satisfied simultane-

ously. This is also true when $\left. \frac{\partial \kappa}{\partial Y} \right|_{\text{COMMON LINE}} = 0$ since

it then follows from eqns (3.1), (3.2) and (3.3) that

$$u(t_0^-) = 0, \text{ and so } u(t_0^-) \neq u(t_0^+)$$

If this result is viewed in terms of the overall velocity field

of these materials $\underline{u} = \underline{u}(\underline{x}, t)$ then

$$\lim_{\underline{x}_2 \rightarrow \underline{x}_0} \underline{u}(\underline{x}_2, t) \neq \lim_{\underline{x}_2 \rightarrow \underline{x}_0} \underline{u}(\underline{x}_2, t)$$

$$\underline{x}_2 : \kappa(\underline{x}_2, t) = 0 \qquad Y < 0$$

where \underline{x}_0 is a point on the common line at time t .

The existence of the left-hand limit presumes a certain amount of smoothness in the motion of the fluid while the right-hand limit is well defined as a consequence of the solid being composed of a rigid material.

In summary, if it is assumed that (i) a material point

on the fluid-fluid interface is mapped onto the common line
in a finite interval of time, (ii) the solid boundary is
rigid and has a planar surface, and (iii) the material point
mapped onto the common line undergoes no motion relative to the
surface once contact has been made, then the velocity field
must be multivalued at the common line. (It will be shown in
Chapter VI that this conclusion also holds for $\frac{\partial \kappa}{\partial \chi} = 0$
at the common line.)

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IV. Kinematical consequences of the Basic Assumption - Bounding surfaces are not necessarily material surfaces.

Over the years it has been common practice for mechanicians, when investigating the motion of a fluid, to solve a boundary value problem for the velocity field. When a bounding surface is present, an appropriate boundary condition (Lord Kelvin, 1848) has been stated in words: "If a fluid mass be in motion, under any conceivable circumstances, its bounding surface will always be such that there will be no motion of the fluid across it." Apparently there had been some controversy at the time as to the correctness of this statement; this is evidenced by a footnote in the above cited paper, which reads in part: "Mr. Stokes [George Gabriel Stokes] is, I believe, the only writer who has given this view [the above quotation] of the subject, all other authors having taken for the condition to be expressed, that a particle [equivalent to a fluid point] which is once in the bounding surface always remains in the bounding surface." Even today the confusion still persists as to the definition of a bounding surface. In this study it is assumed that Lord Kelvin's definition is the appropriate one. Parenthetically, if the second definition is assumed appropriate, then the Basic Assumption (refer to the second section) would be senseless.

Lord Kelvin's (1848) aim was to express this physical concept analytically, i.e. as a mathematical expression in

terms of the velocity field and the equation $F(x, y, z, t) = 0$ giving the position of the bounding surface. This expression could then be used as one of the boundary conditions needed to solve the partial differential equations mentioned above. The analytical expression he derives is as follows:

$$\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0 \quad (4.1)$$

for all (x, y, z, t) which satisfy $F(x, y, z, t) = 0$

This equation is considered to be a mathematical statement which is a necessary consequence of the physical definition of a bounding surface. However, it will now be shown, using the results of the previous section, that equation (4.1) does not necessarily have to be satisfied everywhere on the bounding surface in all physically realistic flow fields. Specifically, it does not have to be satisfied at those points on the bounding surface which coincide with the common line. Here, the point of view is taken that the surface $F = 0$ is the surface of the solid. Since this is a kinematical argument, it doesn't matter if one fluid or two different fluids are bounded by the solid. To derive equation (4.1), Lord Kelvin (1848) proceeds as follows: "To express the fact that every particle of the fluid remains on the same side of the surface, or that there is no *flux* across it, we must find the normal motion of the surface, at any point, in an infinitely small time dt , and equate

this to the normal component of the motion of a neighboring fluid particle during the same time." The normal motion of the surface during the time dt , as given by Lord Kelvin, is clearly

$$- \frac{\partial F}{\partial t} \left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right\}^{-1/2} dt$$

assuming the bounding surface is smooth. However, the normal component of the motion of a fluid particle at the surface is not necessarily

$$\left\{ u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} \right\} \left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right\}^{-1/2} dt \quad (4.2)$$

as given by Lord Kelvin, e.g. this is not necessarily true at the common line. To illustrate this point, let us examine the trajectory of a fluid point which makes contact with the bounding surface at the common line at time $t = t_0$ as shown in figure (4.1). Since $\left. \frac{dY}{dt} \right|_{t=t_0}$ is not well-defined,

$Y = Y(t)$ cannot be represented in the neighborhood of $t = t_0$ as a Taylor series expansion in time, i.e.

$$Y(t) \approx Y(t_0) + \left. \frac{dY}{dt} \right|_{t=t_0} (t - t_0) \quad \text{is not valid.} \quad (4.3)$$

Hence, even though $v(t_0^-) \neq 0$, this does not imply that the fluid point penetrates the surface. To assume that

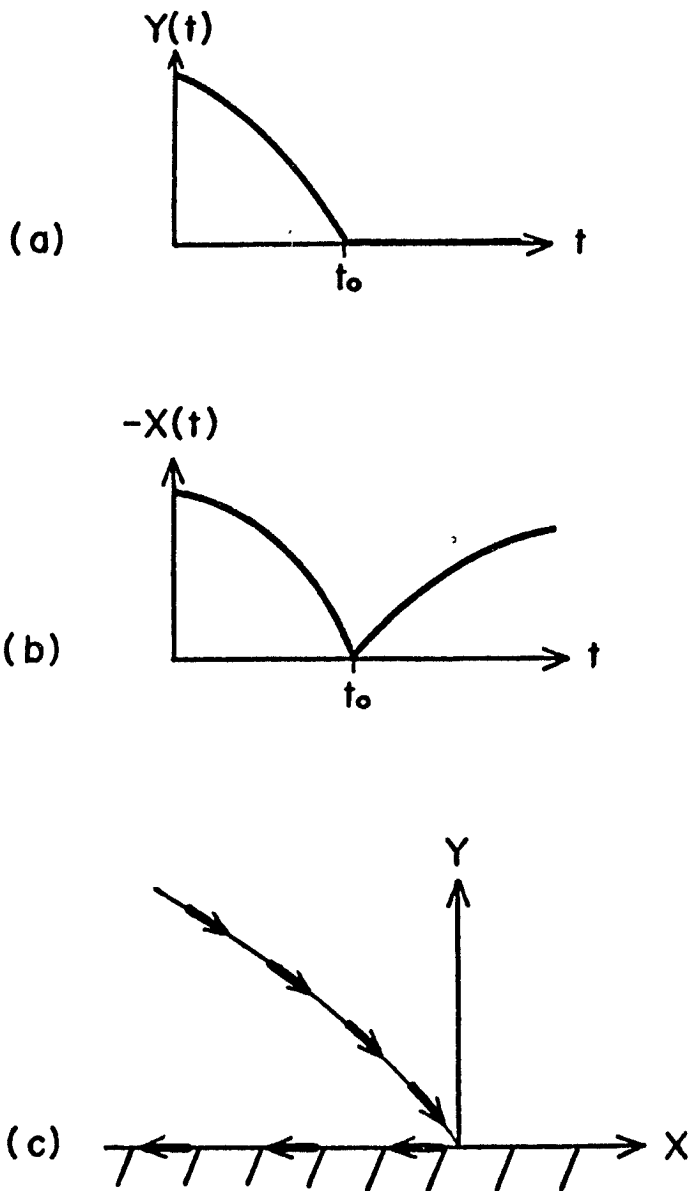


Fig.(4.1) (a) and (b) are the X and Y coordinates of the trajectory at different times respectively. (c) is the projection of the trajectory of the fluid point on the x - y plane at $z=0$.

eqn (4.2) does represent the normal motion of the fluid at the bounding surface is equivalent to assuming that eqn (4.3) is valid. Hence, it has been shown that eqn (4.1) does not have to be satisfied everywhere on a surface in order for the surface to be considered a bounding surface. It does not have to be satisfied at the common line.

It is worth mentioning that eqn (4.1) can be written in an equivalent form, i.e.

$$\underline{u} \cdot \underline{n} = v_n \quad \text{for all points } (x, y, z) \quad \text{on the} \\ \text{bounding surface.}$$

where v_n is the component of the velocity of the surface in the direction \underline{n} , perpendicular to itself. This is the form most commonly seen in the literature. The above result is equivalent to stating that $\underline{u} \cdot \underline{n} = v_n$ need not be satisfied at every point on the bounding surface.

The converse of the above statement is not necessarily true. That is, if $\underline{u} \cdot \underline{n} = v_n$ almost everywhere on the surface $F(x, y, z, t) = 0$, then the surface is not necessarily a bounding surface. Even if the possibility of a common line is eliminated from consideration, the converse, in general, still does not hold unless further assumptions are made about the velocity field. Lord Kelvin was quick to point out this very fact. Eqn (4.1) is only a necessary (not sufficient) condition for a bounding surface.

Truesdell (1951) has investigated sufficient conditions under which the surface $F(x, t) = 0$ is both a bounding surface and a material surface. He has demonstrated that (i) if the velocity field is continuous, (ii) if the surface $F(x, t) = 0$ is differentiable with $|\nabla F| \neq 0, \infty$ and (iii) if the mass of the fluid is conserved without the density ρ being infinite or zero, then the condition

$$\frac{dF}{dt} = 0 \quad \forall x \in F = 0$$

is necessary and sufficient for $F(x, t) = 0$ to be both a bounding surface and a material surface. (If reference is made to the original article, note the difference in notation:

$F(x, t)$ in this study equals $f(x, t)$ in Truesdell's article. $\bar{F}(R, t)$ in this study equals $F(R, t_0, t)$ in Truesdell's article). It will now be shown that due to a restricted definition of a material surface used therein, these results are not totally general and fail to apply to a class of problems that will be described.

Truesdell's analysis will now be described. The position of the surface $F(x, t) = 0$ in the present space x , and in the reference space R will be examined. These two spaces are related by the motion of the material $x = \chi(R, t)$ where $R = \chi^{-1}(x, 0)$. The image of the surface $F(x, t) = 0$ in the R -space is defined to be the surface $\bar{F}(R, t) = 0$ so that

$$F(\underline{x}, t) = F(\underline{\gamma}(\underline{R}, t), t) \equiv \bar{F}(\underline{R}, t) = 0.$$

The speed of propagation C of the surface in the present $\underline{\gamma}$ -space is defined by

$$C \equiv -\frac{\partial F}{\partial t} \left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right\}^{-1/2}$$

The normal speed v_n of the material points instantaneously on $F(\underline{x}, t) = 0$ is defined by

$$v_n \equiv \underline{u} \cdot \left\{ \underline{i} \frac{\partial F}{\partial x} + \underline{j} \frac{\partial F}{\partial y} + \underline{k} \frac{\partial F}{\partial z} \right\} \left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right\}^{-1/2}.$$

This is well-defined since it is assumed that $\underline{u} \in C^0$.

The speed of propagation C_0 of the surface in \underline{R} -space is given by

$$C_0 \equiv -\frac{\partial \bar{F}}{\partial t} \left\{ \left(\frac{\partial \bar{F}}{\partial X} \right)^2 + \left(\frac{\partial \bar{F}}{\partial Y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial Z} \right)^2 \right\}^{-1/2} \quad (4.5)$$

where $\underline{R} \equiv (X, Y, Z)$

Combination of the above three definitions and introduction of the material derivative yields that

$$\begin{aligned} \frac{\partial \bar{F}}{\partial t} = \frac{dF}{dt} &= (v_n - C) \left\{ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right\}^{1/2} \\ &= -C_0 \left\{ \left(\frac{\partial \bar{F}}{\partial X} \right)^2 + \left(\frac{\partial \bar{F}}{\partial Y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial Z} \right)^2 \right\}^{1/2}. \end{aligned} \quad (4.6)$$

A necessary condition for $F(x, t) = 0$ to be a bounding surface is that no material cross it:

$$C = v_n$$

(It has been shown at the beginning of this chapter that this is only valid if $u \in C^0$ which is the case in this analysis).

It follows from eqn (4.6) that $\frac{dF}{dt} = 0$ is a necessary condition for a surface to be a bounding surface. Truesdell attributes the analysis, up to this point, to Lord Kelvin (1848). Truesdell draws an additional conclusion from eqn (4.6); if

$$\left\{ \left(\frac{\partial \bar{F}}{\partial x} \right)^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial z} \right)^2 \right\}^{1/2} \neq 0 \quad \text{and} \quad C = v_n$$

$$\text{then} \quad C_0 = 0$$

In other words, if the surface is a bounding surface (i.e.

$C = v_n$) and if the motion of the material is restricted so that $\left\{ \left(\frac{\partial \bar{F}}{\partial x} \right)^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial z} \right)^2 \right\}^{1/2} \neq 0$, then the speed

of propagation C_0 of the surface in \underline{R} -space is zero.*

* Truesdell shows that

$$\left\{ \left(\frac{\partial \bar{F}}{\partial x} \right)^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial z} \right)^2 \right\} = 0 \iff \frac{\partial(x, y, z)}{\partial(x, y, z)} = 0.$$

Since the equation of conservation of mass can be written in

The difficulty comes when conclusions are inferred from these results.

It is true that a necessary and sufficient condition that $F = \bar{F} = 0$ be a material surface is for $\bar{F}(R, t) = \bar{F}(R)$, but then Truesdell assumes without proof that

$$\bar{F}(R, t) = \bar{F}(R) \Leftrightarrow c_0 = 0 \quad \text{for all choices of reference frame. (4.7)}$$

If this were true, then Truesdell's conclusion would be valid, i.e. that provided $\rho \neq 0, \infty$ and hence that

$$\left\{ \left(\frac{\partial \bar{F}}{\partial x} \right)^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial z} \right)^2 \right\} \neq 0, \text{ then}$$

(footnote continued)

the form

$$\rho(x, t) = \rho_0(R, t) \left\{ \frac{\partial(x, y, z)}{\partial(x, y, z)} \right\}^{-1}$$

then through combination of the above two equations, it follows that

$$\left\{ \left(\frac{\partial \bar{F}}{\partial x} \right)^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)^2 + \left(\frac{\partial \bar{F}}{\partial z} \right)^2 \right\} = 0 \Leftrightarrow \rho = \infty \text{ upon } F = 0$$

provided that $\rho_0 \neq 0, \infty$.

$$\frac{dF}{dt} = 0 \quad \text{on} \quad F=0$$

is both necessary and sufficient for $F=0$ to be a material surface and hence a bounding surface.

However, eqn (4.7) is not true in general. This is demonstrated by the following example. It suffices to look at a two-dimensional problem involving a line of finite length as shown in figure (4.2).

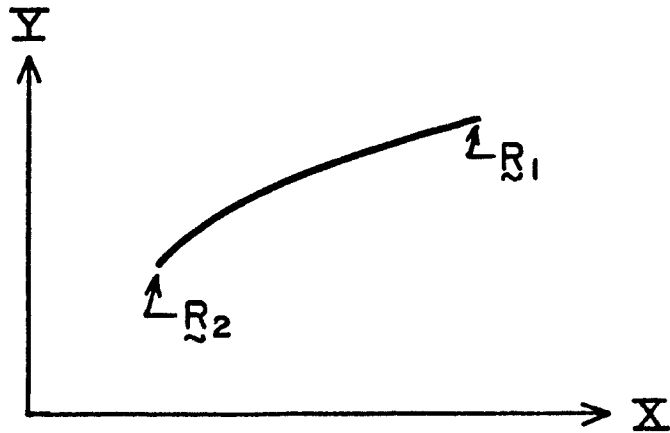


Fig. (4.2) A line of finite length with end points \underline{R}_1 and \underline{R}_2 .

The end points are denoted by \underline{R}_1 and \underline{R}_2 which can be functions of time. Let the line move in such a way so that it extends itself at \underline{R}_1 and recedes at \underline{R}_2 as shown in figure (4.3).

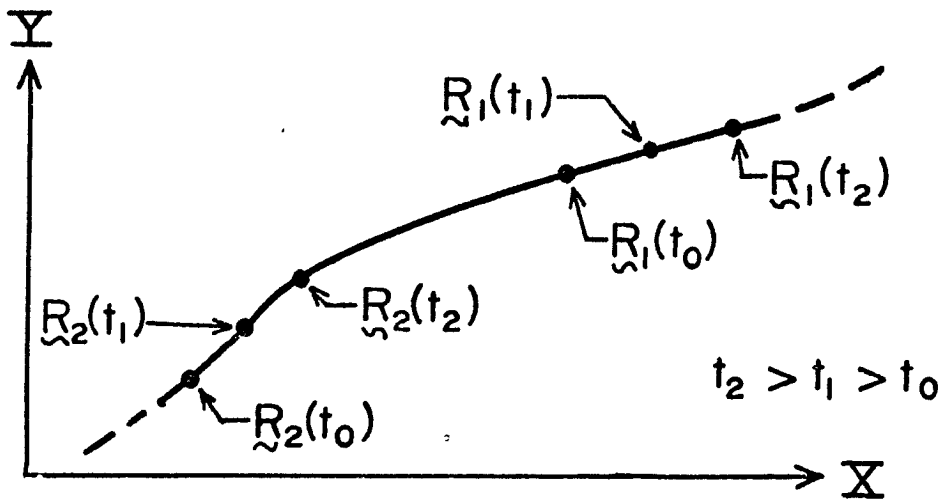


Fig. (4.3) A curve of finite length with end points $R_1 = R_1(t)$ and $R_2 = R_2(t)$.

For this motion $C_0 \equiv 0$ but \bar{F} is a function of time.

This can happen because $\left(\frac{\partial \bar{F}}{\partial X}\right)^2 + \left(\frac{\partial \bar{F}}{\partial Y}\right)^2$ is not

well-defined at the end points $R_1(t)$ and $R_2(t)$ of the line and hence C_0 is not well-defined there either. If

attention is restricted to only those material deformations and surfaces $F=0$ which give rise to surfaces $\bar{F}=0$ having the property that either $\bar{F}=0$ is of infinite extent,

i.e. its perimeter lies at ∞ , or else the perimeter of $\bar{F}=0$ is a material line, then eqn (4.7) is correct and

Truesdell's conclusions are valid. The kinds of flow fields

and surfaces which result in a time dependent surface $\bar{F}=0$ illustrated in the above example are by no means restricted to pathological situations. In fact, this time-dependence in the surface $\bar{F}=0$ occurs in the moving common line as shown in the following example.

Consider a two-dimensional problem involving two deforming materials M_1 and M_2 which form a moving common line on a flat surface as shown in figure (4.4).

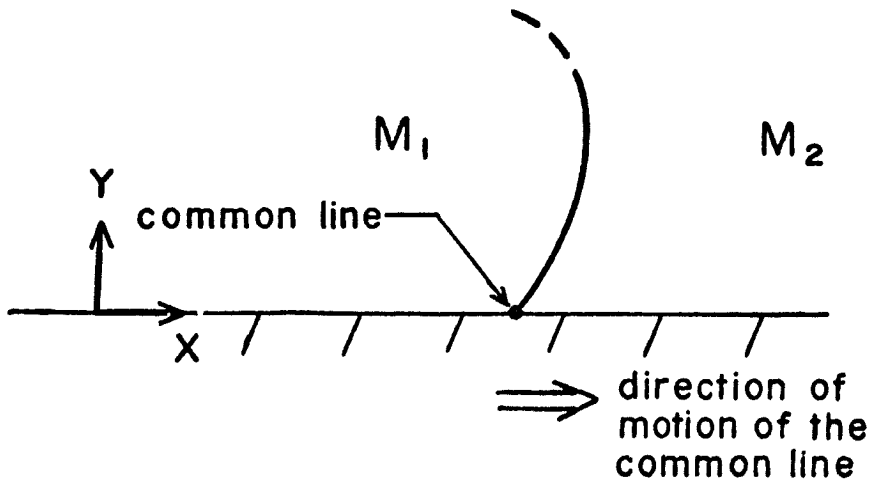


Fig. (4.4) A moving common line formed between materials M_1 , M_2 and a flat bounding surface.

In this case, the equation $F(x_2, t)=0$ has the simple form

$$F(x_2, t) = Y = 0 .$$

At time $t=0$ (arbitrarily chosen) a "picture is taken". The space represented in this picture is the \mathbb{R}^3 -space. It is assumed that the materials undergo a deformation consistent with the Basic Assumption: specifically, the material points on the M_1-M_2 interface are mapped onto the solid surface as shown in figure (4.5). For such a motion the surface $\bar{F}=0$ looks like figure (4.6). To locate the surface $\bar{F}=0$ at time $t=t_3$ one first examines the surface $F=0$ (see figure (4.5)) and identifies the material points which lie on it at this instant in time. For example, at time $t=t_3$ material points 1, 2, and 3 are located on the surface. The "picture" of the system taken at time $t=0$ is returned to a surface through all the material points which are in contact with the solid surface at time $t=t_3$ is drawn; see figure (4.6). In this example, no assumption is made concerning the deformation of the planar solid wall. The assumption that the wall is flat can easily be relaxed.

Hence, Truesdell's theorem does not apply to the moving common line problem. This lack of application is not due to the appearance of the discontinuity in the velocity field at the common line, demonstrated in the previous section. The discontinuity arose as a direct consequence of assuming that the bounding surface is rigid. This can be eliminated by allowing the solid material to deform (while still preserving the no-slip condition). The lack of application is due to the fact

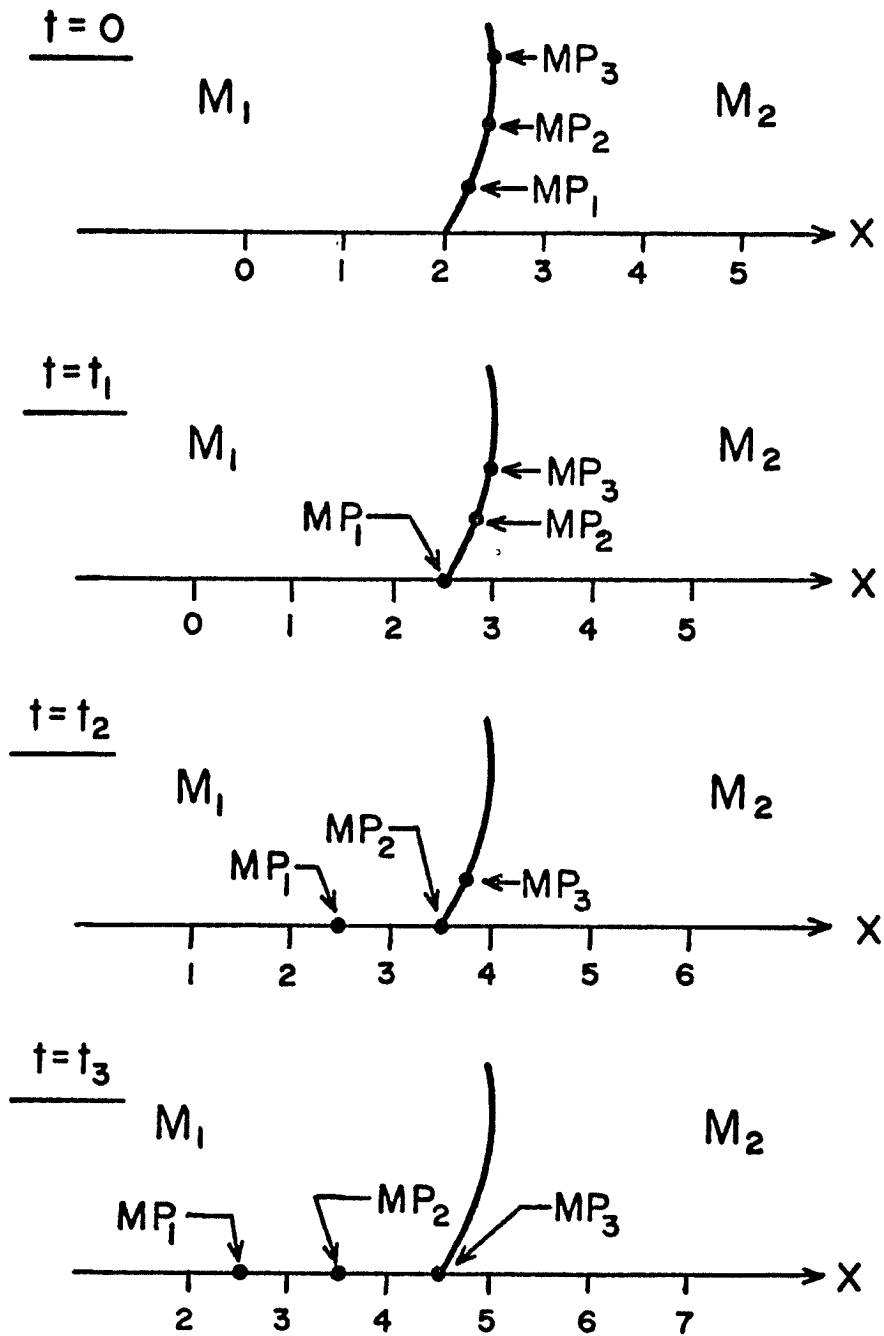


Fig. (4.5) The coordinate system is at rest relative to the solid surface. At time $t=0$ three material points MP_1 , MP_2 and MP_3 are tagged. Their location at subsequent times t_1 , t_2 , t_3 is shown where $0 < t_1 < t_2 < t_3$.

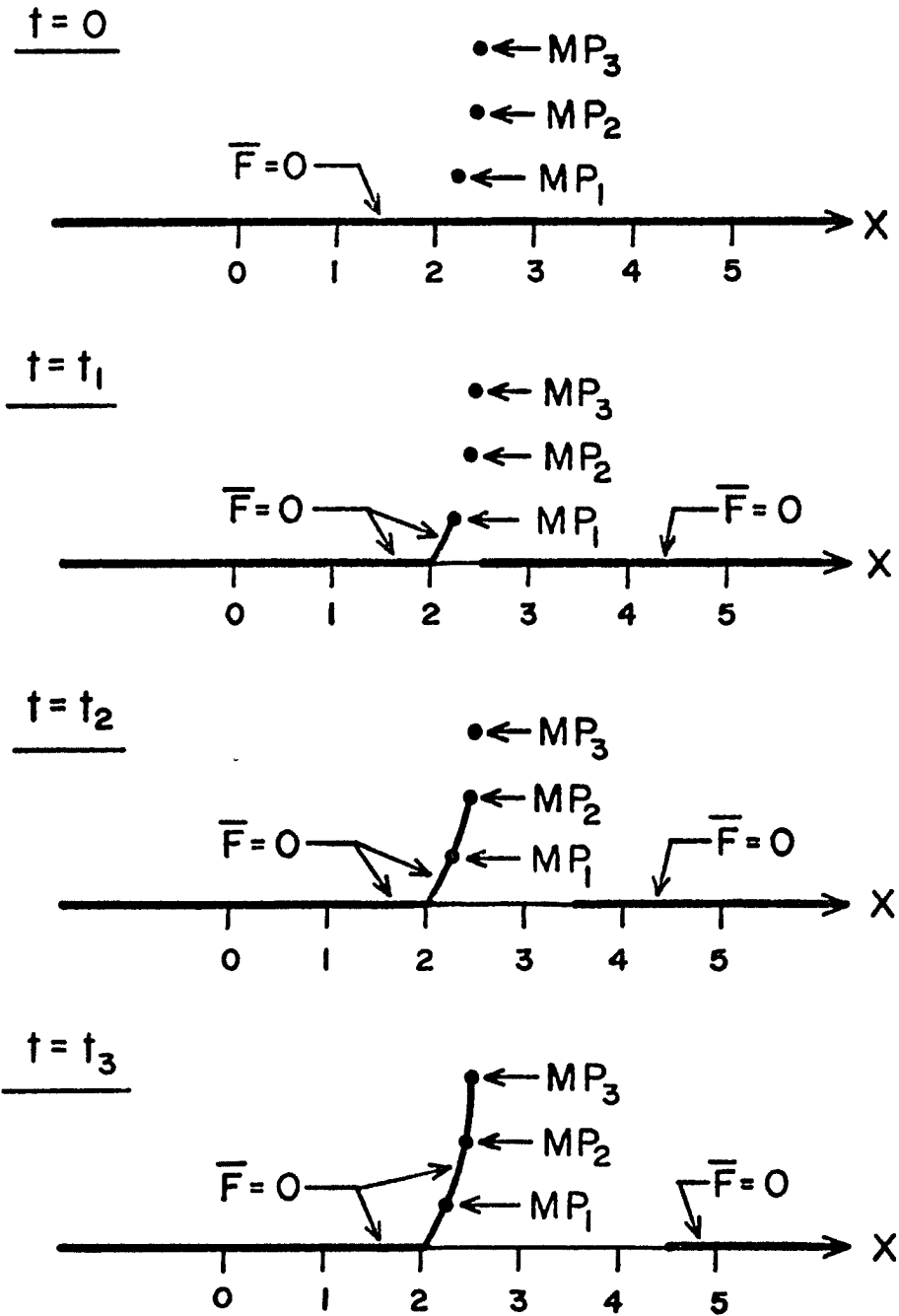


Fig. (4.6) The darkened lines represent the surface $\bar{F} = 0$. All of these figures are drawn in R_2 -space at different times.

that the moving common line (as characterized by the Basic Assumption) gives rise to a function \bar{F} which has the simultaneous properties of being time dependent and $C_0 \equiv 0$ where it is well-defined.

If for some reason one decided to include the fluid-fluid interface in the definition of the "bounding surface", i.e. $F=0$ consists of the surface of the solid and the fluid-fluid interface, then one would still have the situation in which C_0 is not well-defined everywhere on $\bar{F}=0$.

V. Application of the Cauchy-Picard Theorem

The Basic Assumption, as stated in Chapter II, poses restrictions on the form of the velocity field. One restriction is a direct consequence of the requirement that the motion at the common line be non-unique. That is, the material point which is located at R_0 , on the common line, at $t=0$ either "has come from" or "will go onto" the fluid-fluid interface at time $t=t'$ ($t' \neq 0$). Interpreting this in terms of the motion of the material requires that the point $\chi_2(R_0, t')$ lie on the fluid-fluid interface. But the material point located at R_0 at $t=0$ is also a "solid" material point and hence $\chi_2(R_0, t')$ must also lie on the fluid-solid interface. Therefore, the function $\chi_2(R, t)$ which represents the motion of the deforming materials must be non-unique.

If the motion $\chi_2(R, t)$ of a material is known, then the velocity field is obtained as follows:

$$\underline{u} = \underline{\bar{u}}(R, t) \equiv \left. \frac{\partial \chi_2(R, t)}{\partial t} \right|_{R = \text{CONSTANT}}$$

where $\underline{u} = \underline{u}(\chi, t) = \underline{u}(\chi_2(R, t), t) = \underline{\bar{u}}(R, t)$.

Likewise, if the velocity field $\underline{u} = \underline{u}(\chi, t)$ is given, then the motion $\chi_2 = \chi_2(R, t)$ is obtained by solving a system of ordinary (non-linear) differential equations:

$$\frac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t) \quad (5.1)$$

with initial condition $\underline{R} = \underline{x}(\underline{R}, 0)$

It is not surprising that different properties of the velocity field give rise to specific characteristics in the associated motion; in fact, it is not obvious that eqn (5.1) can be solved for any arbitrarily specified velocity field. However, some insight can be obtained from a fundamental theorem in non-linear ordinary differential equations (Ince, 1926; page 62) attributed to different people, the most notable of which are Picard and Cauchy. This theorem states that if in a rectangular domain \mathcal{D} , about the point \underline{R} at $t=0$ defined by $|t| \leq T$ and $|\underline{x} - \underline{R}| \leq \underline{a}$ where $\underline{a} \equiv (a_1, a_2, a_3)$, the velocity field $\underline{u}(\underline{x}, t)$ has the following properties:

(i) The velocity field $\underline{u}(\underline{x}, t)$ is single-valued and continuous in \underline{x} and t . Hence there must exist a positive number M such that $|\underline{u}(\underline{x}, t)| \leq M$.

If h , defined by
$$h \equiv \min\left(\frac{a_1}{M}, \frac{a_2}{M}, \frac{a_3}{M}\right)$$

is less than T then the domain \mathcal{D} is further restricted by

$$|t| < h$$

(ii) The velocity field $\underline{u}(\underline{x}, t)$ satisfies the Lipschitz condition within \mathcal{D} . This means if (\underline{x}_1, t) and (\underline{x}_2, t) are any two points within the domain \mathcal{D} , then there exists three non-negative numbers K_1, K_2, K_3 independent of t , where $K_1 + K_2 + K_3 > 0$ such that

$$|u_i(\underline{x}_1, t) - u_i(\underline{x}_2, t)| < K_i \cdot |\underline{x}_1 - \underline{x}_2|$$

FOR $i = 1, 2, 3$

where $\underline{K} \equiv (K_1, K_2, K_3)$, $\underline{u} \equiv (u_1, u_2, u_3)$ and

$$\underline{u} \cdot \underline{w} \equiv u_1 w_1 + u_2 w_2 + u_3 w_3$$

Then there exists a function $\underline{x}(t)$ which is defined for $|t| < h$ with the following properties:

(iii)

$$\frac{d\underline{x}(t)}{dt} \equiv \underline{u}(\underline{x}(t), t).$$

(iv) The function $\underline{x}(t)$ is a unique and continuous function of t .

(v) $\underline{R} = \underline{x}(0)$

Since a different trajectory, $\underline{x}(t)$ results from a different choice in initial condition, \underline{R} , we can use a slightly different notation and write, $\underline{x} = \underline{x}(\underline{R}, t)$ where

$$\underline{R} = \underline{x}(\underline{R}, 0).$$

A corollary to the above theorem (Ince, 1926; page 68) concerning the dependence of the function $\underline{x}(\underline{R}, t)$ on \underline{R}

is as follows: If the velocity field $\underline{u}(\underline{x}, t)$ satisfies all the conditions in the previous theorem in some domain \mathcal{D} , then the solution $\underline{x} = \underline{X}(\underline{R}, t)$ is uniformly differentiable with respect to \underline{R} when $|t| < h$

We now apply these results to the physical problem of the moving common line. For the situation when the solid bounding surface is modeled as a rigid material with the requirement that the displacing fluids not slip on its surface, then we have seen in Chapter III that the velocity field

$\underline{u}(\underline{x}, t)$ must be discontinuous at the common line. This violates condition (i) in the theorem; hence no conclusions can be drawn. However, for the case when the solid bounding surface is composed of a material that can deform (which relieves the necessity for a discontinuous velocity field), then for the motion of the materials to be non-unique at the common line (this is the Basic Assumption) necessitates the violation of condition (ii). That is, for the Basic Assumption and the desire for a continuous velocity field to be compatible, then the velocity field must not be Lipschitz continuous at the common line.

Parenthetically, it can be shown that if at a fixed ... point in space, \underline{x}_0 , the velocity field is always zero, i.e.

$\underline{u}(\underline{x}_0, t) = 0$ for all t , and that the velocity field obeys conditions (i) and (ii) in the neighborhood of \underline{x}_0 , then the point in space \underline{x}_0 must be a material point. The

trajectory of this material point reduces to $\chi(t) = \chi_0$
and it has the following properties:

$$(a) \quad \frac{d\chi(t)}{dt} = \frac{d\chi_0}{dt} = 0 = \underline{u}(\chi_0, t)$$

FOR $|t| < h$

$$(b) \quad \chi_0 = \chi_0(0)$$

Hence $\chi(t) = \chi_0$ is a solution and in fact the only solution as a consequence of condition (iv). This means that the point at the center of a flow field, which locally can be approximated by a stagnation point flow, is a material point.

VI. To show the existence of an emitting surface from the common line.

The motion of material points, which at some time are located on the fluid-fluid interface, has been investigated. It has been shown in previous chapters by means of qualitative experiments that in some systems it is reasonable to assume material points on the fluid-fluid interface are continually being mapped onto the common line. Once the material points make contact with the bounding surface at the common line, they remain on the surface for subsequent times. The motion of the fluid material points (in the second fluid) which initially are in contact with the bounding surface will now be investigated. As the common line moves forward, it will be shown that these fluid points must be displaced. For the motion described above these fluid points on the fluid-solid interface of the second fluid are mapped into the interior of that fluid. Results of qualitative experiments are presented which suggest the existence of such a flow field.

A material point on the fluid-solid interface has two identities. This is a direct consequence of a material body being a closed set. If a point χ_o , on the interface at time t , is approached from within the fluid (otherwise in any direction), then it is identified as a fluid material boundary point. On the other hand, if the same point χ_o is approached from within the solid, then it is identified as a

solid material boundary point.

In order to describe the mechanics of materials, it is often necessary to deal with scalar-valued functions such as density, viscosity, modulus of elasticity, etc. These scalar-valued functions are usually not well-defined on interfaces. However, if it is desired to assign a value to a scalar-valued function at a fluid material boundary point, then it is usually given the one sided (fluid) limiting value, i.e.

$$\rho_f(x_0, t) \equiv \lim_{\substack{x \rightarrow x_0 \\ x \in \text{FLUID}}} \rho(x, t)$$

where ρ is, for example, the density function which is well-defined within the interior of the fluid, and $\lim_{\substack{x \rightarrow x_0 \\ x \in \text{FLUID}}}$ is the one sided limit which is independent of path as long as this path lies within the fluid. The existence of ρ_f , or rather the existence of the "one sided limit" requires a certain amount of smoothness in the function $\rho(x, t)$ in the interior of the fluid in a neighborhood of x_0 . Likewise, the density function, or for that matter, any scalar-valued function, evaluated at the solid material boundary point located at x_0 at t is given as:

$$\rho_s(x_0, t) \equiv \lim_{\substack{x \rightarrow x_0 \\ x \in \text{SOLID}}} \rho(x, t) .$$

In a similar fashion the motion, or the trajectory, of fluid material boundary points can be specified as the limit of the motion of neighboring fluid material points. That is, it shall be assumed that the trajectory $\chi_f(t)$ of the fluid material boundary point located at R_o at $t=0$ is given by:

$$\chi_f(t) \equiv \lim_{\substack{R \rightarrow R_o \\ R \in \text{FLUID}}} \chi_f(R, t) \quad \text{for all time.}$$

where R_o is on the fluid boundary, $\chi_f(R, t)$ is the motion for the entire system and $R_o = \chi_f(R, 0)$. The trajectory of the solid material boundary point $\chi_s(t)$ also located at R_o at $t=0$, is given by:

$$\chi_s(t) \equiv \lim_{\substack{R \rightarrow R_o \\ R \in \text{SOLID}}} \chi_s(R, t) \quad \text{for all time.}$$

Again, it is assumed the motion of the materials are sufficiently smooth so that the one-sided limits given in the above definitions make sense. In addition, it is required that the trajectories $\chi_f(t)$ and $\chi_s(t)$ satisfy the following:

$$\frac{d\chi_f(t)}{dt} = v(\chi_f(t), t),$$

and
$$\frac{d\chi_s(t)}{dt} = u(\chi_s(t), t),$$

where $u(\underline{x}, t)$ is the velocity field of the entire system which is usually assumed to be continuous.

Attention is focused on the three-dimensional, unsteady moving common line. The motion is viewed from a frame of reference moving with a point (not a material point) on the common line. By a moving common line, it is specifically meant that the line formed by the intersection of the fluid-fluid interface and the solid bounding surface consists of different solid material points at different times. Without loss of generality, and for definiteness, attention is focused on one segment of the common line in which (a) material points on the fluid-fluid interface are mapped onto the common line, (b) solid material points on the bounding surface which are moving towards the common line are those on the fluid #1-solid interface (This is just a definition of fluid #1.). This particular segment of the common line, $\underline{x} = \underline{f}(s, t)$, is denoted by $0 \leq s \leq S$ with the segment possessing properties (a) and (b) for t :

$|t| \leq T$. It shall be shown that if:

- (i) It is assumed the trajectory of the fluid #1 material points in contact with the bounding solid is given by:

$$\underline{x}_{t_1} = \lim_{\substack{R \rightarrow R_0 \\ R \in \text{FLUID \#1}}} \underline{\psi}(R, t) \quad \text{for all time.}$$

where R_0 is located at the fluid #1-solid interface at $t = 0$.

(ii) The mass of fluid #1 is conserved with no sources or sinks and with the density function finite everywhere including on its bounding surface.

(iii) The velocity field for the whole system is Lipschitz continuous in the domain excluding any small neighborhood of the common line,

then the fluid points on the fluid #1-solid interface are mapped onto the common line and then into the interior of fluid #1. (We are referring here to the segment of the common line described above with properties (a) and (b).) The two-dimensional case is illustrated in figure (6.1).

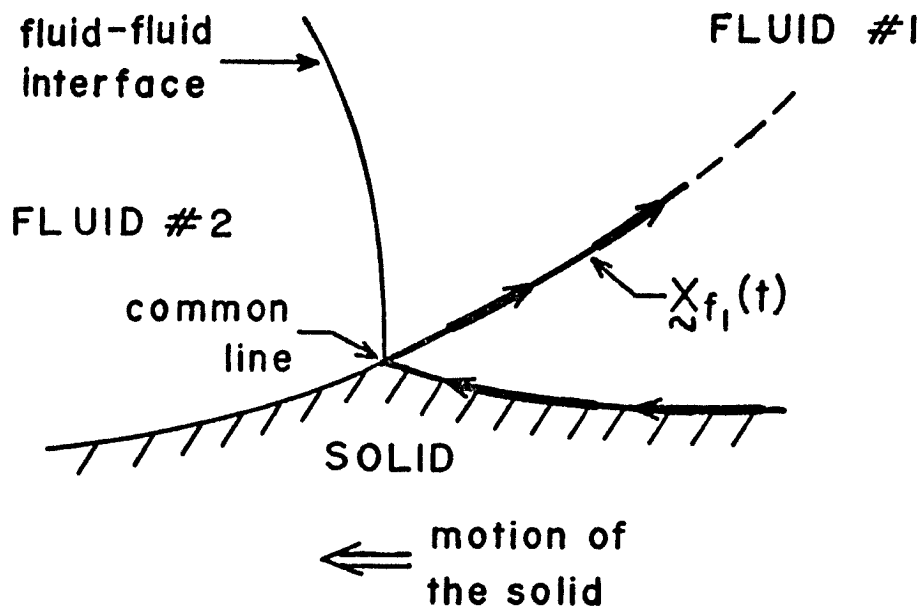


Fig. (6.1) Two-dimensional motion with frame of reference defined such that the common line is still.

To demonstrate the above, let us investigate the

trajectory of a fluid #1 material point and a solid material point located at \underline{R}_0 at $t=0$. This is a point on the fluid #1-solid interface which is not on the common line. The initial position of the trajectories $\underline{X}_f(t)$ and $\underline{X}_s(t)$ is indeed \underline{R}_0 since:

$$\underline{X}_f(0) = \lim_{\substack{\underline{R} \rightarrow \underline{R}_0 \\ \underline{R} \in \text{FLUID \#1}}} \underline{X}(\underline{R}, 0) = \lim_{\substack{\underline{R} \rightarrow \underline{R}_0 \\ \underline{R} \in \text{FLUID \#1}}} \underline{R} = \underline{R}_0,$$

$$\underline{X}_s(0) = \lim_{\substack{\underline{R} \rightarrow \underline{R}_0 \\ \underline{R} \in \text{SOLID}}} \underline{X}_s(\underline{R}, 0) = \lim_{\substack{\underline{R} \rightarrow \underline{R}_0 \\ \underline{R} \in \text{SOLID}}} \underline{R} = \underline{R}_0.$$

In the above equations $\lim_{t \rightarrow 0} \lim_{\underline{R} \rightarrow \underline{R}_0} \underline{X}(\underline{R}, t)$ is interchanged with $\lim_{\underline{R} \rightarrow \underline{R}_0} \lim_{t \rightarrow 0} \underline{X}(\underline{R}, t)$. By assumption (iii) and the fact that \underline{R}_0 at $t=0$ is not on the common line, the theorem of Picard and Cauchy discussed in the previous section gives that the motion $\underline{X}(\underline{R}, t)$ of the system is a continuous function of t and \underline{R} in some closed domain about \underline{R}_0 and $t=0$. This is a sufficient condition that the limits may be interchanged (Townsend, 1928; p.113). It also follows from the Picard-Cauchy theorem, that outside some neighborhood of the common line the solution is unique. Hence it follows that

$$\underline{X}_f(t) = \underline{X}_s(t)$$

as long as χ_s is not on the common line. By making assumption (iii), it necessarily implies that the flow field must obey the no slip condition. Implicit in property (b) concerning the segment of the common line under investigation is that there exists a time T_1 associated with R_0 and S_0 such that

$$\chi_s(R_0, T_1) = \xi(s_0, T_1)$$

where $0 < s_0 < S$ and $0 < T_1 < T$. It follows that:

$$\chi_s(T_1) \in \text{common line.}$$

It is now shown that if it is required that

$$\chi_{s_1}(t) = \chi_s(t) \quad \text{for } t > T_1,$$

then the density function ρ is unbounded at the common line; this contradicts assumption (ii). Without loss of generality, since it is assumed that the common line is in motion, there exists a time T_2 : $T_1 < T_2 < T$

such that

$$|\chi_s(T_2) - \chi_s(T_1)| > D > 0 \tag{6.1}$$

AND HENCE

$$|\chi_{s_1}(T_2) - \chi_{s_1}(T_1)| > D > 0.$$

By assumption (i) it follows that for any $\epsilon > 0$ there exists an $\delta > 0$ (possibly dependent on t) such that

$|\underline{x}_s(T_2) - \underline{x}_s(\underline{R}, T_2)| < \epsilon$ for \underline{R} within fluid #1 and satisfying $|\underline{R} - \underline{R}_0| < \zeta$. In other words, all the fluid #1 material points, \underline{R} , within a distance of ζ from \underline{R}_0 , are mapped to within a distance ϵ of $\underline{x}_s(T_2)$. However, since $\underline{x}_s(T_2)$ is located on the portion of the bounding surface which is referred to as the fluid #2-solid interface, means that the images of all \underline{R} satisfying $|\underline{R} - \underline{R}_0| < \zeta$ and within fluid #1 must lie identically on the bounding surface. Hence, the mass M of fluid #1 within $|\underline{R} - \underline{R}_0| < \zeta$ at $t=0$, i.e.

$$M \equiv \int_{\substack{|\underline{R} - \underline{R}_0| < \zeta \\ \underline{R} \in \text{FLUID \#1}}} \rho(\underline{x}, 0) dV,$$

must be mapped into a region, at time $t = T_2$, which has zero volume, implying that ρ is not finite at the common line.

Since an unbounded density function is not permitted by assumption (ii), there are only two other alternatives as to the location of $\underline{x}_s(t)$ for $t > T_1$; (1) the material point is mapped onto the fluid-fluid interface, (2) the material point is mapped into the interior of fluid #1. The first alternative cannot occur due to property (a); fluid points on the fluid-fluid interface are mapped onto the common line. Hence, the only possible path for the fluid #1 point is into the interior of fluid #1.

It has been shown for a system of materials with a segment which possesses properties (a) and (b), whose motion is restricted to obey assumptions (i) - (iii) and with a deforming solid which obeys eqn (6.1), that the fluid points on the bounding surface must be mapped into the interior of the fluid.

Following an identical line of reasoning, it can be shown that a fluid #1 material point located on the fluid-fluid interface and mapped onto the common line must then be mapped into the interior of fluid #1. The set of all material points in the interior of fluid #1 which originally were located on the fluid-fluid interface, form a surface (totally in the interior of fluid #1) which intersects the bounding surface at the common line. Likewise, a surface within fluid #1 exists which consists of material points originally located on the fluid #1-solid interface. These surfaces emitted from the common line are illustrated in figure (6.2) for the special case of two-dimensional steady motion.

The trajectory $\bar{\chi}_{s_1}(t)$ is that of a fluid #1 material point initially on the fluid-fluid interface. The parts of $\bar{\chi}_{s_1}(t)$ and $\chi_{s_1}(t)$ extending into the interior of fluid #1 coincides respectively with the first and second surface described above. It is also possible that the two surfaces coincide as shown in figure (6.3).

In the unsteady case these surfaces may "wave" around.

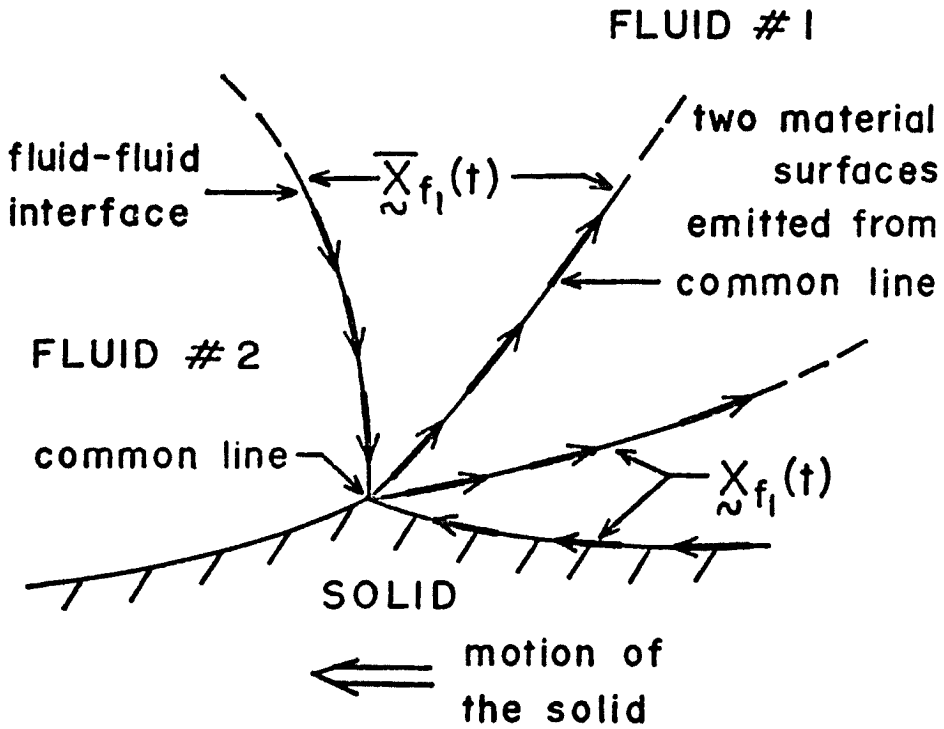


Fig. (6.2) Surfaces emitted from the common line consisting of fluid #1 material points originally located on the surface of fluid #1. This is a steady two-dimensional motion.

The material points MP_1 , MP_2 , MP_3 and MP_4 are located at three successive instances in time $t_1 < t_2 < t_3$, in figure (6.4).

These emerging surfaces could be visualized by dyeing a piece of fluid #1 which has, as shown in figure (6.5a) part of its boundary coinciding with the fluid #1-solid interface. As time progresses, the dyed fluid #1 material points on the fluid #1-solid interface travel towards the common line, as

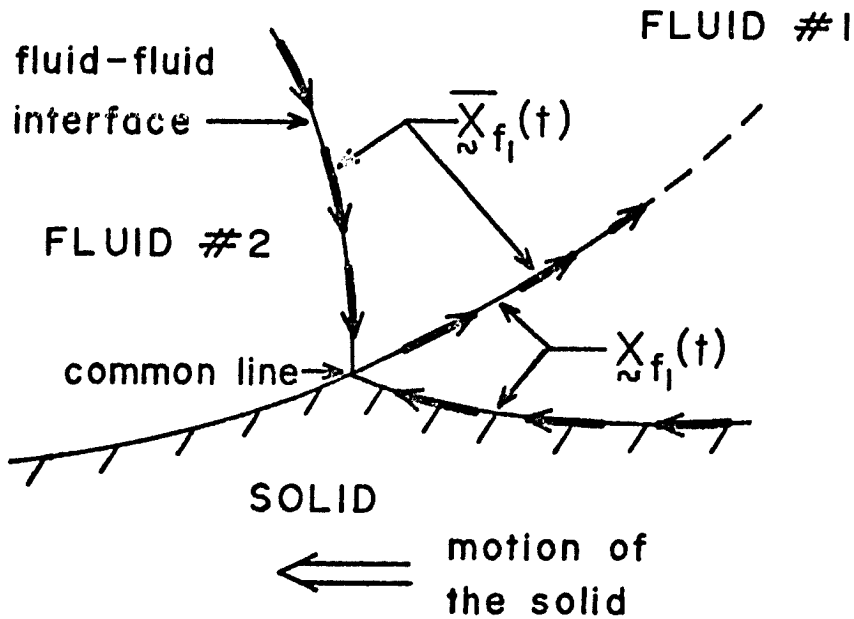


Fig. (6.3) Only one emitted surface from the common line.

seen in figure (6.5b). At some later time, material points MP_1 and MP_2 have been mapped into the interior of fluid #1, as seen in figure (6.5c). The boundary of the dye extending from MP_1 to the common line is a piece of the emerging surface. Finally, the whole piece of dye is within fluid #1, as shown in figure (6.5d). The part of the bounding of the dye which connects material points MP_1 , MP_2 , MP_3 and MP_4 is also part of the sought after surface.

The same procedure can be used to visualize the "upper" emerging surface within fluid #1 consisting of material initially located on the fluid-fluid interface. A portion of

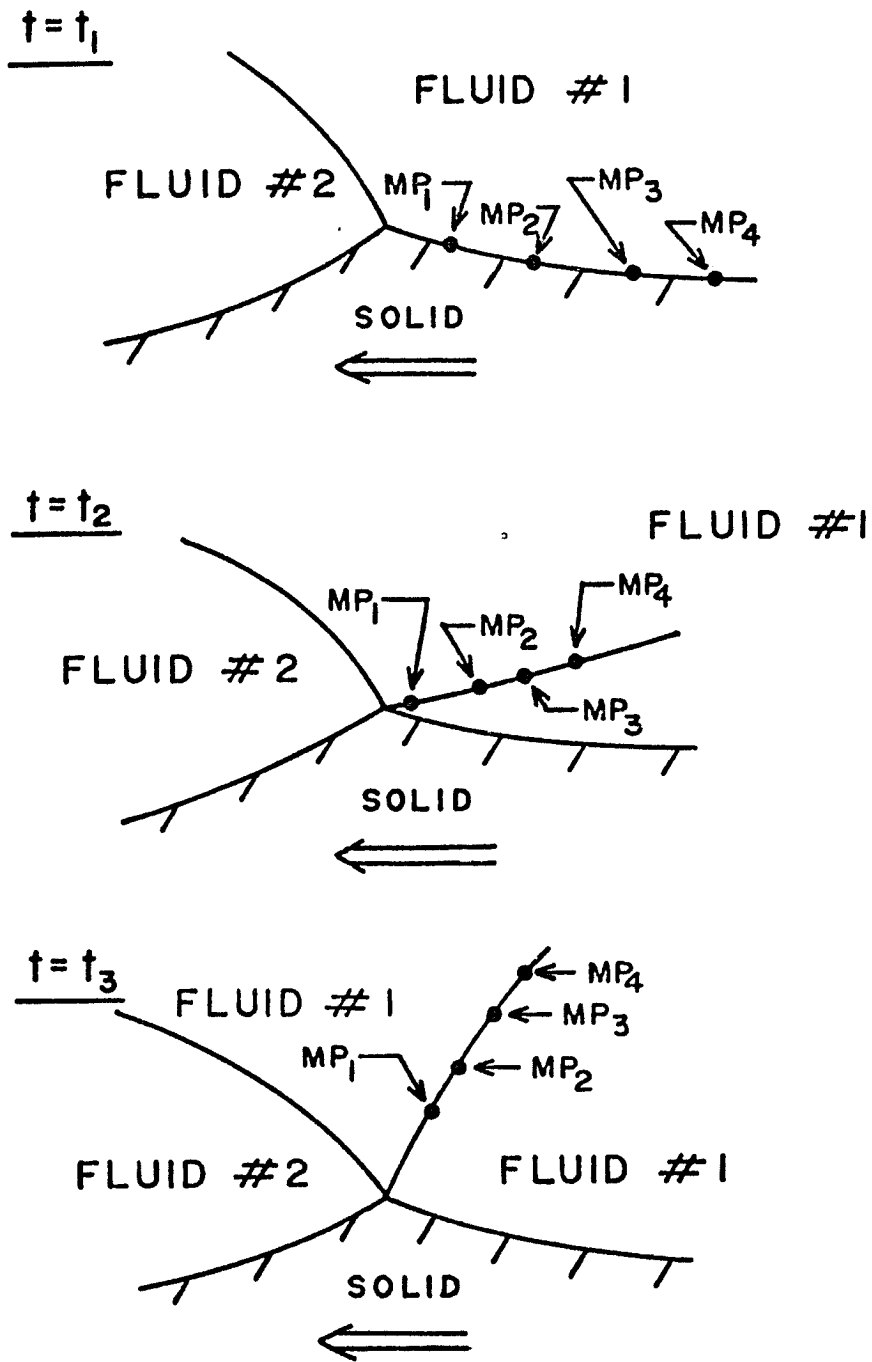


Fig. (6.4) The location of four fluid #1 material points MP_1 , MP_2 , MP_3 and MP_4 at three times $t_1 < t_2 < t_3$ for an unsteady motion.

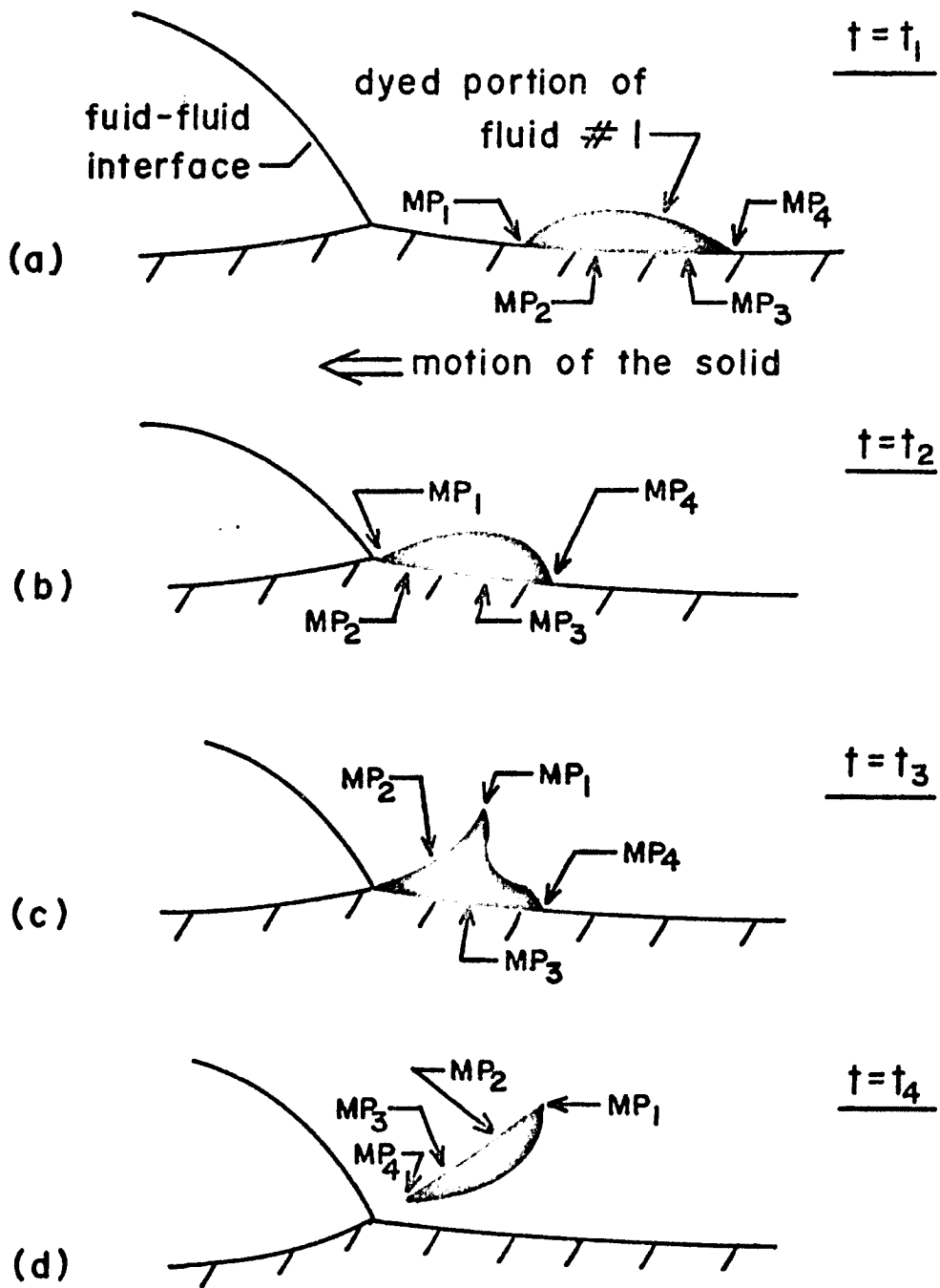


Fig. (6.5) The trajectory of a dyed portion of fluid #1 initially in contact with the solid bounding surface, $t_1 < t_2 < t_3 < t_4$.

fluid #1 which has, as shown in Figure (6.6a), part of its boundary coinciding with the fluid-fluid interface, is dyed. The position of the dye at subsequent times is given in figures (6.6b), (6.6c), (6.6d). In figure (6.6c) the portion of the boundary of the dye extending from material point MP_1 to the common line is also a piece of the emerging surface. In figure (6.6d) the part of the boundary of the dye connecting MP_1 , MP_2 , MP_3 , and MP_4 is, again, a piece of the emerging surface.

Experiments similar to the above have been performed to observe these surfaces being emitted from the common line. The system consists of a plexiglass container with a rectangular base, silicone oil (Union Carbide L-45, $\gamma = 100$ CTSK), and a mixture of about 60% water and 40% methyl alcohol. The common line formed by the alcohol water mixture, the silicone oil and the plexiglass surface moves in any direction depending on the configuration of the entire system. Refer to figure (6.7) for a picture of the system.

A mixture of alcohol, water and McCormick's food dye is injected onto the plexiglass-alcohol and water interface. The dye is not disturbed for a while to allow it to diffuse very close to the bounding surface. The right end of the tank (the end which is occupied by oil) is slowly raised causing the portion of the common line stretching across the bottom of the tank to move towards the alcohol-water fluid. When the common

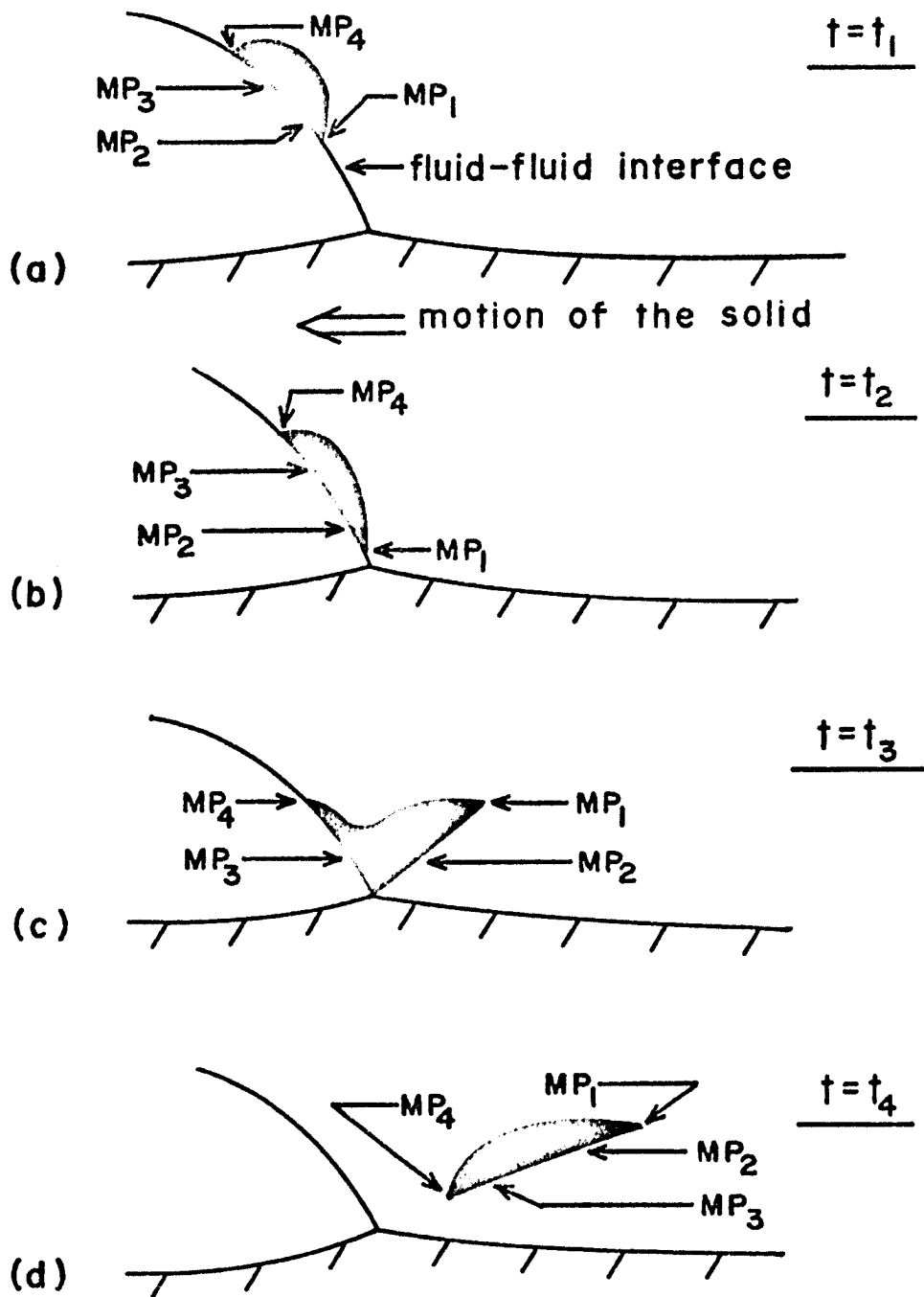
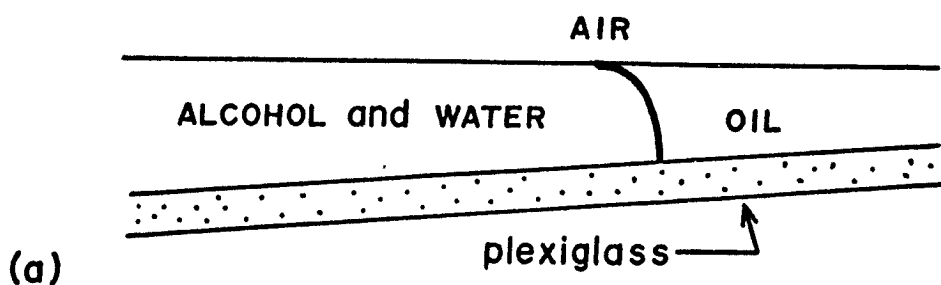


Fig. (6.6) The trajectory of a dyed piece of fluid #1 initially in contact with the fluid-fluid interface,

$$t_1 < t_2 < t_3 < t_4.$$



(a)



(b)

Fig. (6.7) Equilibrium configuration for the alcohol and water-oil system. (a) is an illustration from the side view. (b) is a picture of the same view.

line appears to be "touching" the dye, the dye has a wedge shape. Finally, the dye is located in the interior of the alcohol and water fluid as shown in figure (6.8).

The above experiment is repeated but this time the dye is applied to the alcohol and water-oil interface. As the dye is ejected from a hypodermic needle, it spreads out into a disk

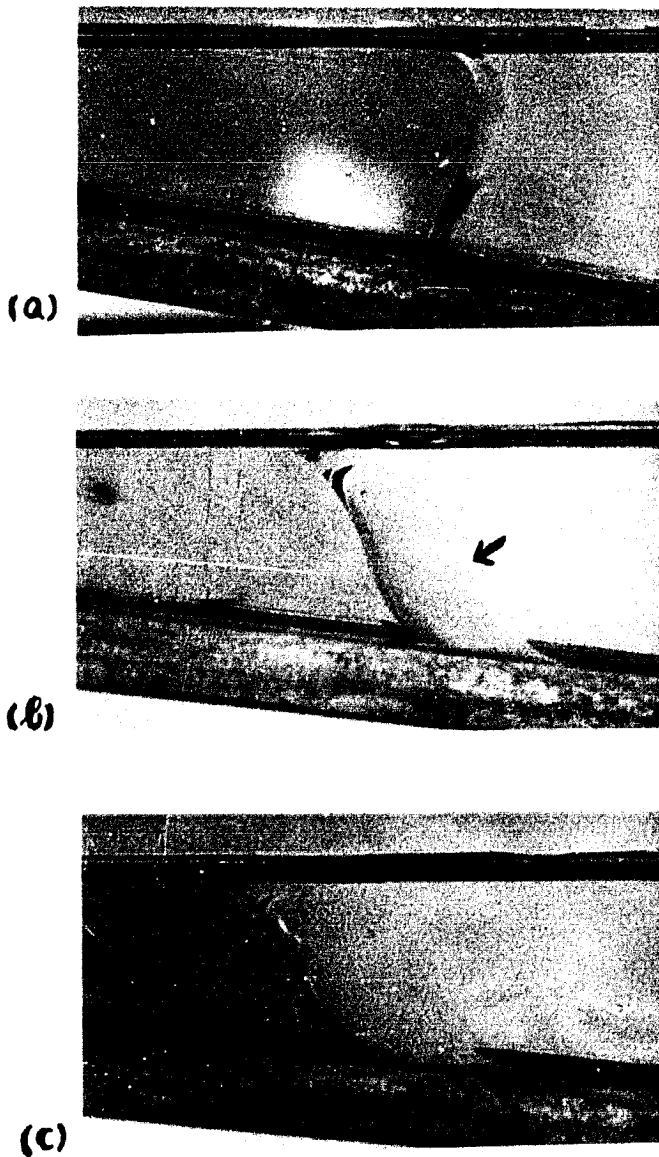


Fig. (6.8) The fluid-fluid interface in (a) is relatively flat, i.e. perpendicular to the page. In (b) and (c) the interface bends. The arrow in (b) points to the location of the common line on the front face of the container. The arrow in (c) points to that portion of the fluid-fluid interface which is located midway between the front and back of the container. The portion of the dye facing the interface in (c) appears "fuzzy" due to the fact that dye is bent; consequently in (c) we are observing only its outer edges.

shaped configuration. The spreading seems to be caused by surface tension gradients due to the dye locally lowering the surface tension.

As the oil end of the container is raised and as the portion of the common line on the bottom of the tank moves toward the alcohol-water fluid, it is observed that the dye on the interface moves toward the common line. After a short while, a sheet of dye emerges from the common line region into the alcohol-water fluid. The bottom side of this sheet can be seen in figure (6.9).

It is interesting to note that both emerging surfaces coincide in this experiment, i.e. the flow is similar to that illustrated in figure (6.3). It has also been confirmed that the motion in the oil is a "rolling" type of motion as illustrated by the glycerine in a similar previous experiment (Chapter II).

In the beginning of this chapter, it was demonstrated that a consequence of points on the fluid-fluid interface being mapped onto the common line is that ejecting surfaces must exist. In an identical way, one can show that the existence of injecting surfaces is a consequence of common line points being mapped onto the fluid-fluid interface. The system described above has been examined with the common line moving towards the oil. This is achieved by simply lowering the oil end of the tank. In this experiment, dye is applied to the

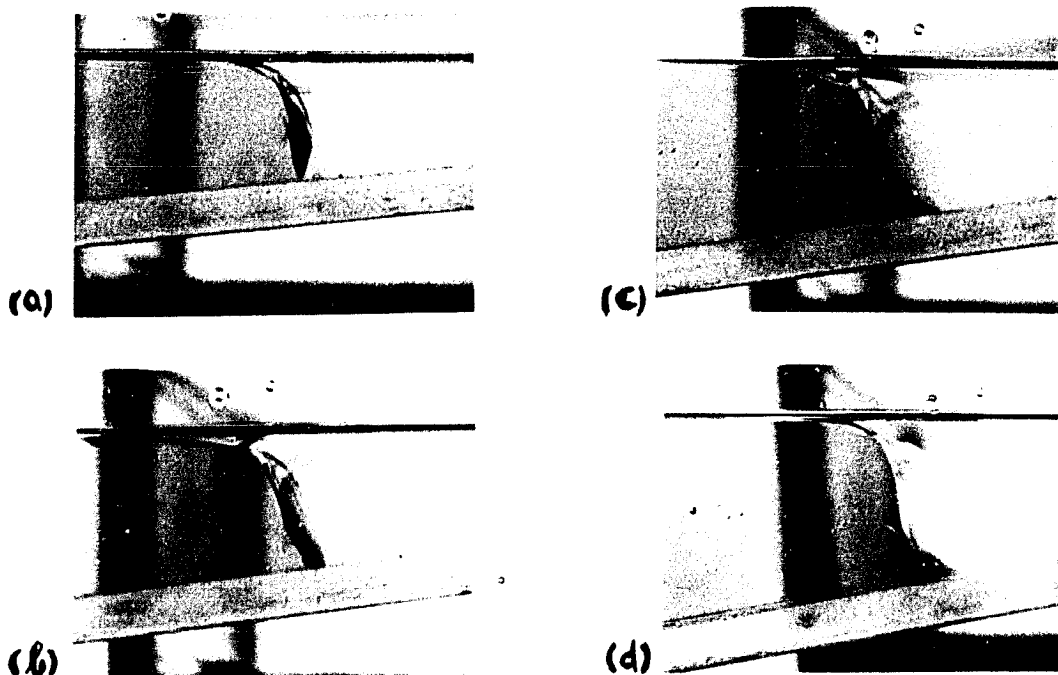


Fig. (6.9) Due to the bend in the interface, we can see only half of the "disk" like initial configuration of the dye in (a). In (b) we begin to see the emitted surface. Again, due to the bend in the fluid-fluid interface, the bottom side of the emitted surface is quite distinct in (c) at the location midway between the front and rear of the container. At positions closer to the front of the container, the emitted surface of dye looks like a dark curtain rising from the bottom of the fluid-fluid interface, as shown in (c) and (d).

neighborhood of the common line in the alcohol-water fluid; refer to figure (6.10).

As the oil end of the container is lowered, the common line moves toward the oil and the dye deforms as shown in figure (6.11).

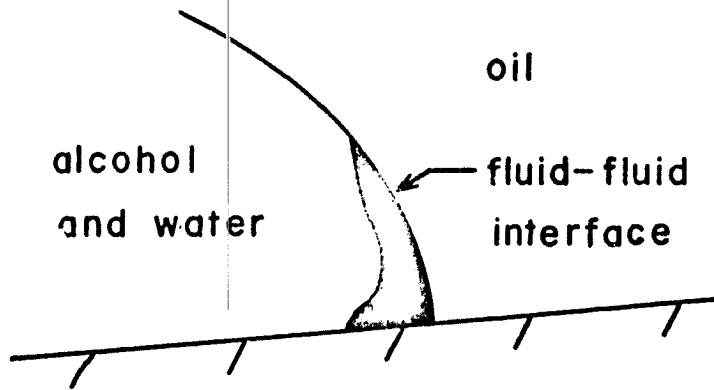


Fig. (6.10) Two-dimensional illustration of the experiment. The dye mark consists of alcohol, water and food dye.

The dye distinctly divides into two parts. The configuration of the piece of dye on the fluid-fluid interface varied for different runs of the experiment. These results are consistent with the flow field produced by one injecting surface at the common line; refer to figure (6.12).

In light of the existence of the ejecting or injecting surfaces, the nature of the image $\bar{F}=0$ of the bounding surface $F=0$ (surface of the solid) in \mathcal{R}_2 -space (refer back to Chapter IV) is examined. The function \bar{F} satisfies the following equation:

$$F(\underline{x}, t) = F(\underline{\chi}(\underline{R}, t), t) = \bar{F}(\underline{R}, t) \quad (6.2)$$

where $\underline{\chi}(\underline{R}, t)$ is the motion of the entire system and

$\underline{R} = \underline{\chi}(\underline{R}, 0)$. It is assumed without loss of generality

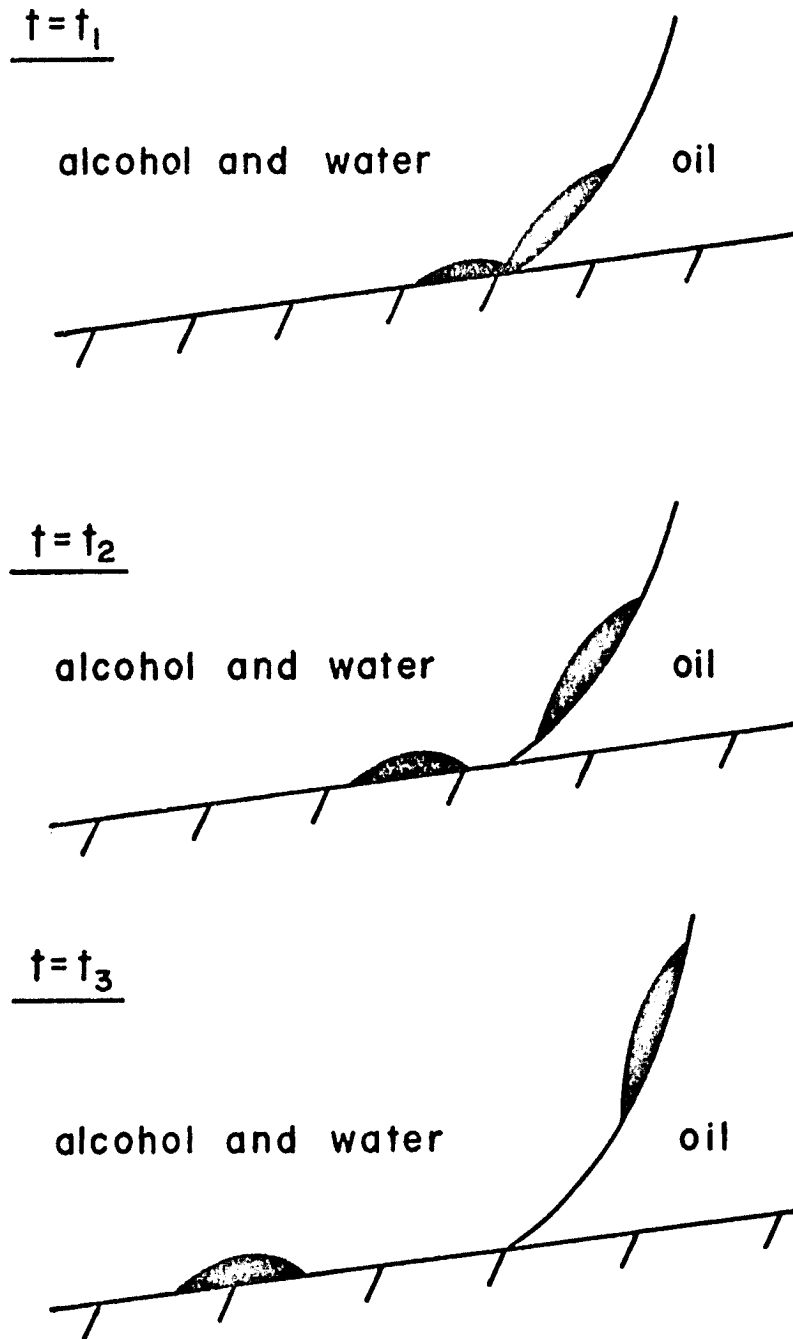


Fig. (6.11) The Deformation of originally one piece of alcohol and water dye as the common line moves towards the oil.

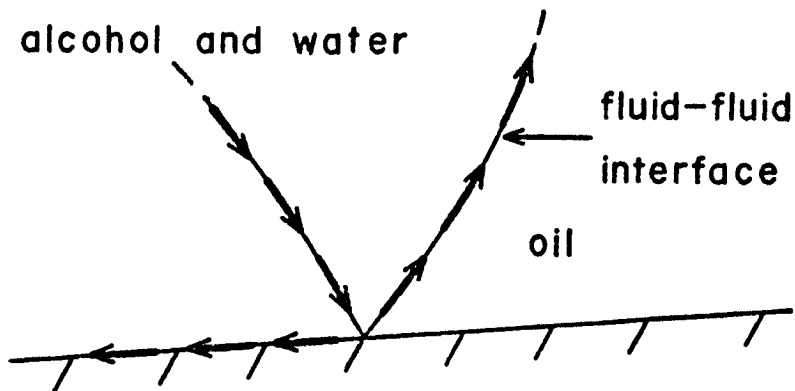


Fig. (6.12) The frame of reference is at rest with respect to the common line. This is an example of an injecting surface.

that the solid is flat, hence $F(x, t) = Y$ and that the motion is two-dimensional and steady. It has been seen that the trajectory of a material point "on" the bounding surface is not unique. Hence the meaning of eqn (6.2) is ambiguous. Which trajectories should be used to get the surface $\bar{F} = 0$? If the trajectories of the solid material boundary points is used, then $F = Y = 0$ for all time implies $\bar{F} = Y = 0$ for all time, where $R = (X, Y)$. The surface $\bar{F} = Y = 0$ is independent of time and hence a material surface; indeed, it is a solid material surface. On the other hand, if the trajectory of the fluid material boundary points is used, then an $\bar{F} = 0$ is obtained as shown in figure (4.6) in chapter IV. This figure is reproduced here for convenience. The material

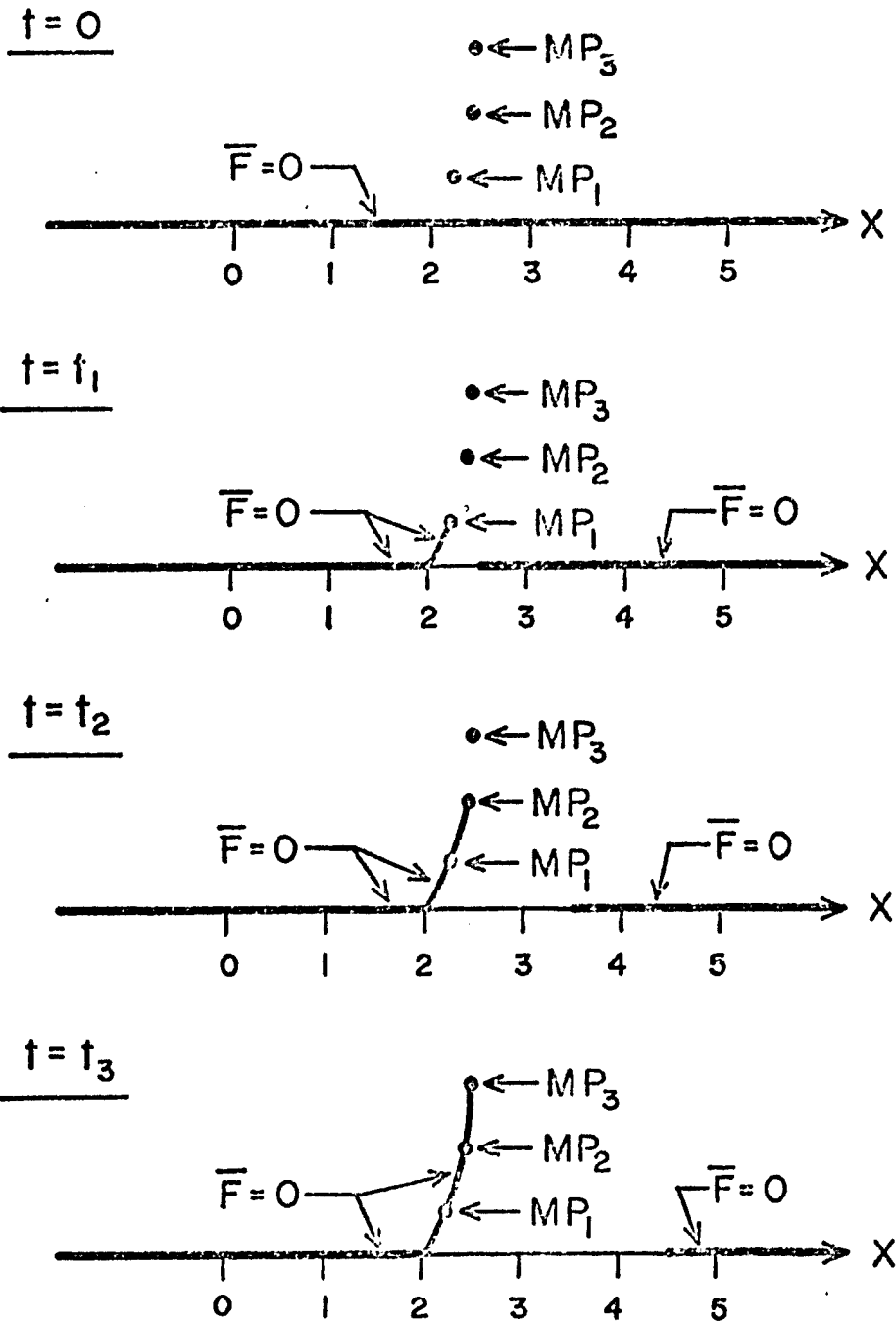


Fig. (4.6) The darkened lines represent the surface $\bar{F} = 0$. All of these figures are drawn in R_2 -space at different times.

points which are located on $Y=0$, $2.5 \leq X \leq 4.5$ at $t=0$ are on the bounding surface. However, at time $t = t_3$, they no longer constitute part of the bounding surface (They are not a part of the surface $\bar{F}=0$). It is this material which composes part of the surface being ejected from the common line. This gives rise to a function \bar{F} which is time dependent and hence not a material surface. Even if the definition of $\bar{F}=0$ is extended to include both the fluid-fluid interface and the surface of the solid, \bar{F} is not a material surface. This is a direct consequence of the existence of a surface ejecting from the common line.

The necessity for the existence of at least one ejecting surface from the moving common line has been demonstrated. This surface can be visualized by means of dye. However, it is also possible for other theoretical models of the moving common line to explain the observations of the trajectory of the dye in the previous demonstrations. Consider the following example of a two-dimensional steady motion as shown in figure (6.13). The bounding surface is rigid and flat and moves with constant speed U_0 . The material points on the fluid-fluid interface move in the same direction as the solid surface and approach a velocity of $(U_0, 0)$ as $\gamma \rightarrow -\infty$, i.e. the interfacial material is moving at the same speed as the bounding surface. It is further assumed that the density of fluid #1 is constant. It is now shown that such a flow gives rise

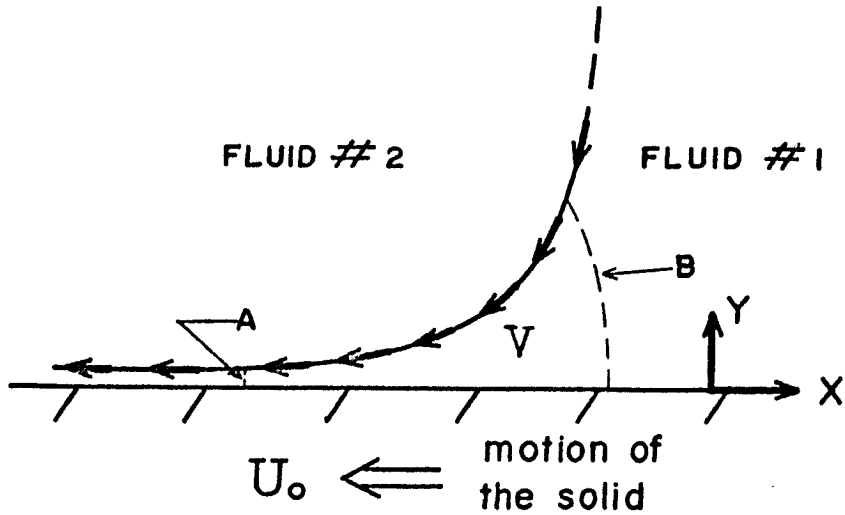


Fig. (6.13) Surface A and B extend from the fluid-fluid interface to the solid bounding surface. The enclosed volume is V . The frame of reference is moving at such a velocity so that the configuration of the fluid-fluid interface appears stationary.

to a surface within fluid #1 which might look like an emitted surface from "far away" but, in fact, is quite different.

Conservation of mass requires that for any stationary volume

V that

$$\int_{\partial V} \underline{u} \cdot \underline{n} \, ds = 0 \quad (6.3)$$

where ∂V is the boundary of V and \underline{n} is its outward normal. V is chosen to be the volume lying between the solid-fluid interface and the fluid-fluid interface and within two surfaces joining these two surfaces, as shown in figure

(6.13) above. (V lies within the dotted lines). If the flow across surface A is roughly U_0 , then

$$\int_A \underline{u} \cdot \underline{n} \, ds \approx U_0 S_A$$
 where S_A is proportional to the area of A . It is found that there must be an influx

of volume near the solid surface due to the no slip condition.

There also must be an influx of volume into V in the neighborhood of the fluid-fluid interface due to the assumed motion

on the interface. If $U_0 S_A$ is less than the sum of these influxes, then there must be a reflux out of B so that

eqn (6.3) is satisfied. One possible motion of fluid #1 is

illustrated in figure (6.14). Such a flow gives rise to a

surface within fluid #1, \mathcal{S} , which may cause the dye markings

in the previously described experiments, as observed "from far

away", to deform in a similar fashion as in figures (6.5) and

(6.6). However, the surface \mathcal{S} is conceptually very much

different from a surface emitted from the common line. The

purpose of this example has been to emphasize that the obser-

vations in figures (6.8) and (6.9) are not prima facie evidence

of the existence of an emitted surface because they can be ex-

plained by means of other theoretical models of the common

line region.

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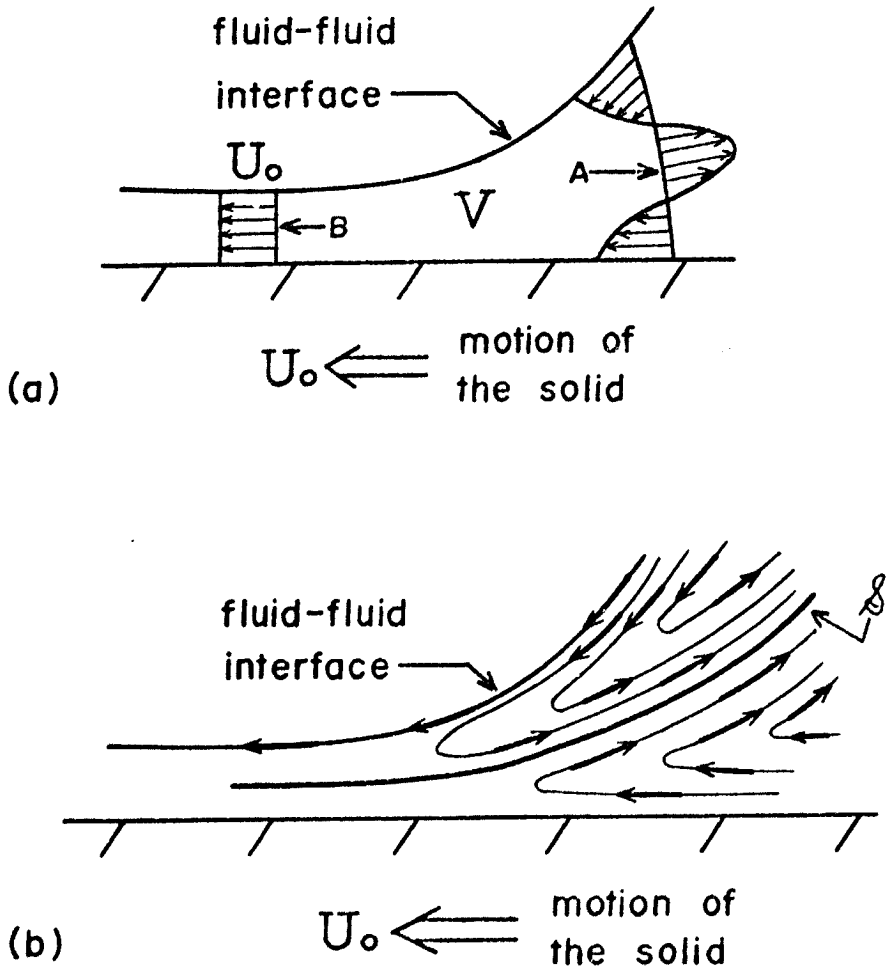


Fig. (6.14) This is a magnified view of the system. In (a) the component of the velocity normal to the surface A and B is indicated. In (b) the trajectory of fluid #1 material points are shown.

VII. To show that the forces exerted by certain classes of fluids are unbounded in the neighborhood of the common line.

It has been shown in Chapter III that the Basic Assumption and the no slip condition on a rigid bounding surface necessarily gives rise to a discontinuous velocity field at the common line. It is easily seen that this type of discontinuous velocity field must necessarily possess gradients which are unbounded at the common line. If the fluids have constitutive relations (an equation which gives the explicit dependences of the stress tensor) which involve the strain rate tensor (the symmetric part of the gradient of the velocity field), then there exists a possibility that the stress tensor must also be unbounded at the common line. This must occur, for example, if the fluids are Newtonian. An unbounded stress tensor, in itself, is not worrisome. There exist physical situations in which the modeling of a force distributed over a small area by a force acting at a point or a line (This implies an unbounded stress tensor.) makes good sense. By "good sense" it is meant that the mathematical model predicts the values of physically measurable quantities, such as the displacement fields, velocity fields and forces, which agree well with experiment. However, it is also possible that an unbounded stress tensor may give rise to an unbounded force. Such a mathematical theory then may be considered physically unrealistic, at least in the region in which this occurs.

In this section the following assumptions are made:

- (A) The velocity is discontinuous at the common line (not necessarily resulting from a rigid bounding surface).
- (B) The fluids are incompressible.
- (C) The motion is two-dimensional and steady.
- (D) The velocity field is represented by a specific form (This is detailed in the analysis later).

Then there exists an angle $\bar{\theta}$ such that $I(\zeta)$, defined by

$$I(\zeta) \equiv \int_0^{\zeta} \gamma \Big|_{\theta=\bar{\theta}} dr$$

and

$$\gamma \equiv \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta}$$

is unbounded for any $\zeta > 0$. The quantities u and v are the radial and azimuthal components of the velocity field (see figure (7.2) for a definition of the coordinate system).

For the special case of a Newtonian fluid the term γ , is proportional to $(\underline{T} \hat{\theta}) \cdot \hat{r}$, which is a component of the stress tensor \underline{T} . The integral $I(\zeta)$ then is proportional to a force exerted by the fluid (This is explained more explicitly after the domain is defined and assumptions (i) - (x) concerning u and v are given.). Hence unbounded I implies unbounded forces within the fluid.

The applicability of the results of the following

demonstration is by no means restricted to Newtonian fluids, since it is not assumed that the velocity field satisfies the Navier-Stokes equation. For the case when the fluid-fluid interface is modeled as a region of finite thickness, still following the Basic Assumption as stated at the beginning of Chapter III, then in the neighborhood of the common line the fluid will most likely be non-Newtonian. It still may be possible that unbounded $I(\gamma)$ gives rise to unbounded forces. However, to ascertain this, a particular constitutive relation of the fluid must be assumed.

When the fluid-fluid interface is modeled as a surface, boundary conditions are usually specified concerning the jump in the stress tensor across the interface. No assumptions are made on these boundary conditions in the analysis. The results are independent of such concepts as surface tension, surface viscosity, surface concentration of any surfactant or a "dirty interface", etc.

Attention is focused on one of the two displacing fluids. A bounded domain \mathcal{D} (closed set) is defined which lies entirely within one of the displacing materials. Its boundary $\partial\mathcal{D}$, is the union of three piecewise smooth curves $\partial\mathcal{D}_1$, $\partial\mathcal{D}_2$, and $\partial\mathcal{D}_3$ which have the following properties:

(a) The segment of the boundary $\partial\mathcal{D}_1$ coincides with a portion of the fluid-fluid interface, with one of its end points coinciding with the common line.

(b) The segment of the boundary $\partial\mathcal{D}_2$ coincides with a portion of the fluid-solid interface, with one of its end points coinciding with the common line. The geometry is shown in figure (7.1). θ_1 is restricted to the range $0 < \theta_1 < \pi$.

(c) The segment of the boundary $\partial\mathcal{D}_3$ is any single-valued smooth curve joining the end points of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$.

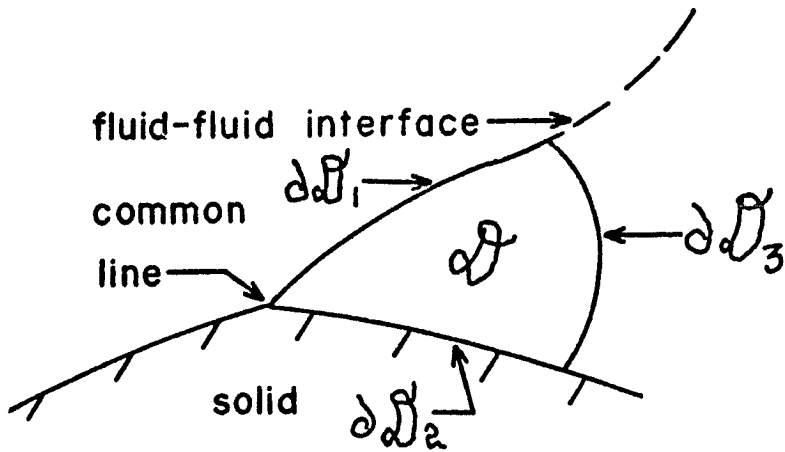


Fig. (7.1) The frame of reference is at rest with respect to the common line.

A polar coordinate system is used which has its origin at the common line. The angle θ , which appears as one of the two ordered pairs of real numbers (r, θ) representing the position vector of a point in space, is the angle formed between the position vector and the tangent vector to $\partial\mathcal{D}_2$.

at the common line as shown in figure (7.2).

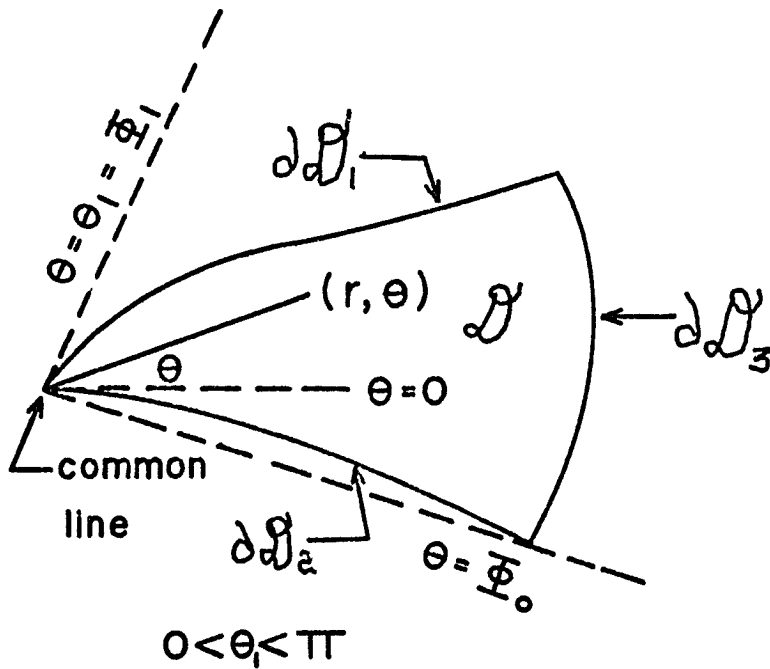


Fig. (7.2) Definition of coordinate system. The angles ϕ_0 and ϕ_1 are defined such that for any point $(r, \theta) \in \partial D$ we have $\phi_0 \leq \theta \leq \phi_1$. The angle θ is formed between the tangent to ∂D_1 and the common line and the tangent to ∂D_2 at the common line.

The following assumptions are made on $u = u(r, \theta)$ and $v = v(r, \theta)$ which represent the radial and azimuthal components of the velocity field, respectively:

(i) u and v are bounded in D .

(ii) u and v are $C^{(3)}$ in $D - C(\epsilon)$

for any $\epsilon > 0$. The symbol $C(\epsilon)$ denotes the open set of points lying within the circle of radius ϵ ,

centered at the origin.

(iii) For any fixed θ in the interval $[\Phi_0, \Phi_1]$, the functions u and v are absolutely continuous functions of r . (For the definition of an absolutely continuous function, see appendix 1.2).

It can be shown (appendix 1.3) that the above assumptions imply the existence of two bounded functions $f(\theta)$ and $g(\theta)$ where:

$$f(\theta) \equiv \lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} u(r, \theta)$$

$$g(\theta) \equiv \lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} v(r, \theta)$$

for $0 < \theta' < \theta_1$. For certain geometries of \mathcal{D} there exists the possibility that the functions f and g are well-defined over the closed interval $0 \leq \theta' \leq \theta_1$. The functions $\bar{u} = \bar{u}(r, \theta)$ and $\bar{v} = \bar{v}(r, \theta)$ are defined for all $(r, \theta) \in \mathcal{D}$ as follows:

$$\bar{u}(r, \theta) \equiv u(r, \theta) - f(\theta)$$

$$\bar{v}(r, \theta) \equiv v(r, \theta) - g(\theta)$$

(iv) $\frac{\partial \bar{u}}{\partial \theta}$ and $\frac{\partial \bar{v}}{\partial \theta}$ exist and are $C^{(1)}$ in $\mathcal{D} - C(\epsilon)$ for any $\epsilon > 0$.

(v) The limits of f and g as $\theta \rightarrow \theta_1$, and $\theta \rightarrow 0$ exist (finite) and are defined as $f(\theta_1)$, $f(0)$, $g(\theta_1)$ and $g(0)$ where:

$$f(\theta_1) = \lim_{\substack{\theta \rightarrow \theta_1 \\ \theta < \theta_1}} f(\theta) \quad ; \quad f(0) = \lim_{\substack{\theta \rightarrow 0 \\ \theta > 0}} f(\theta)$$

$$g(\theta_1) = \lim_{\substack{\theta \rightarrow \theta_1 \\ \theta < \theta_1}} g(\theta) \quad ; \quad g(0) = \lim_{\substack{\theta \rightarrow 0 \\ \theta > 0}} g(\theta)$$

(vi) The following equalities hold:

$$f(\theta_1) = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} u(r, \theta) \quad ; \quad f(0) = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} u(r, \theta)$$

$$g(\theta_1) = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} v(r, \theta) \quad ; \quad g(0) = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} v(r, \theta).$$

$$(vii) \quad \lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \frac{\partial \bar{u}}{\partial \sigma} = \frac{\partial}{\partial \theta} \left\{ \lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \bar{u} \right\} \quad \text{For } 0 < \theta' < \theta_1.$$

By definition of \bar{u} , $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \bar{u} = 0$. Hence, this

assumption can be written as

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \frac{\partial \bar{u}}{\partial \theta} = 0, \quad 0 < \theta' < \theta_1.$$

Likewise,

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \frac{\partial \bar{v}}{\partial \theta} = \frac{\partial}{\partial \theta} \left\{ \lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \bar{v} \right\} \quad \text{FOR } 0 < \theta' < \theta_1.$$

By definition of \bar{v} , it follows that

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \frac{\partial \bar{v}}{\partial \theta} = 0 \quad \text{FOR } 0 < \theta' < \theta_1.$$

(viii) The fluid is incompressible, which implies

$$\frac{\partial v}{\partial \theta} + r \frac{\partial u}{\partial r} + u = 0$$

(ixa) $\underline{u} \cdot \underline{n} = 0$ on $\partial \mathcal{D}_1$, and on $\partial \mathcal{D}_2$ not including the origin. The vector \underline{n} is perpendicular to the tangent vectors of the curves $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_2$.

Since the segments $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_2$ of the boundary are considered smooth with well-defined tangents as $r \rightarrow 0$, we have that

$$\lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} \underline{u} \cdot \underline{n} = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} v \quad \text{AND,}$$

$$\lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} \underline{u} \cdot \underline{n} = \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} v.$$

As a consequence of assumption (ixa) it follows that

$$\lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} v = 0$$

and

$$\lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} v = 0$$

Using assumption (iv) an alternative form for assumption (ix₂)

can be written:

$$g(\theta_1) = 0 \quad \text{AND} \quad g(0) = 0.$$

(ixb)

$$\lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_1}} \underline{u} \cdot \underline{\gamma} \neq \lim_{\substack{r \rightarrow 0 \\ (r, \theta) \in \partial \mathcal{D}_2}} \underline{u} \cdot \underline{\gamma}$$

where $\underline{\gamma}$ is a vector field tangent to the curves $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_2$. This gives a discontinuous velocity at the common line.

Using assumption (iv) an alternative form for assumption (ixb)

can be written:

$$f(\theta_1) \neq f(0)$$

where the geometry is restricted so that $0 < \theta_1 < \pi$.

$$(x) \quad \lim_{\substack{r \rightarrow 0 \\ \theta = \theta_1}} r \frac{\partial \bar{u}}{\partial r} = 0 \quad \text{FOR} \quad 0 < \theta' < \theta_1$$

It can be shown (appendix 1.3) for a neighborhood of $r = 0$ on $\theta = \theta_1$ that if $\frac{\partial \bar{u}}{\partial r}$ is monotonic without bound

or if $\left| \frac{\partial \bar{u}}{\partial r} \right|$ is bounded, then

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} r \frac{\partial \bar{u}}{\partial r} = 0 \quad \text{For } 0 < \theta' < \theta_1.$$

It shall now be shown that if a velocity field satisfies assumptions (i) through (x) in a domain described in (a), (b) and (c), then there exists an angle $\bar{\theta}$: $0 < \bar{\theta} < \theta_1$, where

$$I(\zeta) = \int_0^\zeta \left\{ \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right\} \Big|_{\theta = \bar{\theta}} dr$$

is unbounded for any $\zeta > 0$. As mentioned at the beginning of this section for the case when the fluid is Newtonian,

$$\tau = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \quad \text{is proportional to } (\underline{T} \hat{e}) \cdot \hat{r},$$

a component of the stress tensor \underline{T} (\hat{r} is a unit vector in the radial direction and \hat{e} is a unit vector in the azimuthal direction). The integral $I(\zeta)$ is then proportional to the tangential component of the force exerted by the fluid outside material body B , on the portion of its boundary which is located at $\theta = \bar{\theta}$ and $0 < r < \zeta$; refer to figure (7.3). Hence, for a Newtonian fluid, an unbounded $I(\zeta)$ implies unbounded forces.

To demonstrate this, the expressions for u and v which are obtained from the definitions of \bar{u} and \bar{v}

$$u = f(\theta) + \bar{u}(r, \theta)$$

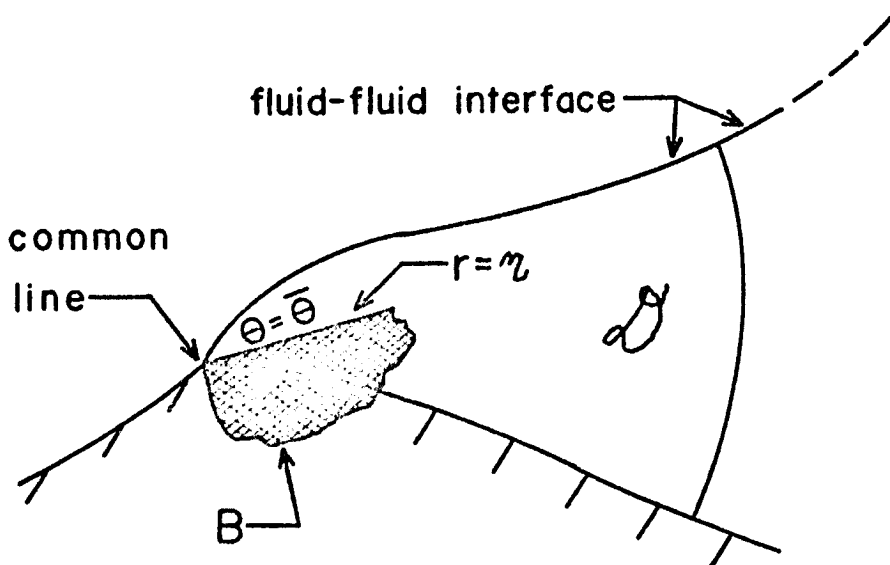


Fig. (7.3) The shaded area represents the body B .

and
$$v = g(\theta) + \bar{v}(r, \theta) ,$$

are substituted into the equation in assumption (viii), the incompressibility condition. This gives:

$$\frac{dg}{d\theta} + \frac{\partial \bar{v}}{\partial \theta} + r \frac{\partial \bar{u}}{\partial r} + f(\theta) + \psi(r, \theta) = 0. \quad (7.1)$$

The derivatives exist as a consequence of assumptions (ii) and (iv). Taking the limit of eqn (7.1) along the ray $\theta = \theta'$ where $0 < \theta' < \theta_1$, and using assumption (x)

$$\frac{dg}{d\theta} + f(\theta) = 0 \quad \text{for } 0 < \theta < \theta_1, \quad (7.2)$$

is obtained. Since $f(\theta)$ is continuous and well-defined over the closed interval $0 \leq \theta \leq \theta_1$, this implies that $\frac{df}{d\theta}$ must be well-defined over the same closed interval for the assumptions to be compatible.

The above expressions for u and v are substituted into the equation for γ :

$$\gamma = \frac{1}{r} \left\{ \frac{df}{d\theta} - g + \frac{\partial \bar{u}}{\partial \theta} - \bar{v} \right\} + \frac{\partial \bar{v}}{\partial r} .$$

Since \bar{v} is an absolutely continuous function of r for fixed θ (assumption (ii) and definition of v), the Radon-Nikodym theorem implies that

$$\bar{v}(\gamma, \theta) = \int_0^\gamma \frac{\partial \bar{v}}{\partial r} dr .$$

Hence the contribution of $\int_0^\gamma \frac{\partial \bar{v}}{\partial r} dr$ to the integral

$I(\gamma)$, is finite.

Consequently, in order for the integral $I(\gamma)$ to be finite for all $0 \leq \bar{\theta} \leq \theta_1$, it is necessary that:

$$\frac{df}{d\theta} - g(\theta) + \frac{\partial \bar{u}}{\partial \theta}(r, \theta) - \bar{v}(r, \theta) = o(1) \quad \text{as } r \rightarrow 0-$$

along a ray. It has been established, by definition of

\bar{v} , that $\lim_{\substack{r \rightarrow 0 \\ \theta = \bar{\theta}}} \bar{v} = 0$ and by assumption (vii) that

$\lim_{\substack{r \rightarrow 0 \\ \theta = \bar{\theta}}} \frac{\partial \bar{u}}{\partial \theta} = 0$, and that f and g are independent

of r . Hence, it follows that

$$\frac{df}{d\theta} - g = 0 \quad \text{for } 0 \leq \theta \leq \theta_1, \quad (7.3)$$

Assumptions (ii) and (iv) permit the differentiation of eqn (7.2) with respect to θ :

$$\frac{d^2g}{d\theta^2} + \frac{df}{d\theta} = 0 \quad 0 \leq \theta \leq \theta_1, \quad (7.4)$$

If eqn (7.3) is substituted into eqn (7.4), then

$$\frac{d^2g}{d\theta^2} + g = 0 \quad 0 \leq \theta \leq \theta_1,$$

The solution of this equation is:

$$g(\theta) = A \sin \theta + B \cos \theta \quad \text{for } 0 \leq \theta \leq \theta_1.$$

The alternative form of assumption (ixa), i.e. $g(0) = g(\theta_1) = 0$ implies that

$$g(\theta) = 0 \quad \text{for } 0 \leq \theta \leq \theta_1 < \pi$$

Eqn (7.2) then gives:

$$f(\theta) \equiv 0 \quad \text{for } 0 \leq \theta \leq \theta_1.$$

However, this contradicts assumption (ixb) which states that:

$$f(0) \neq f(\theta_1) .$$

Hence it has been shown that there exists an angle $\bar{\theta}$ such that $\int_0^{\bar{\theta}} r |dr$ is unbounded for any $\bar{\theta} > 0$ where $0 \leq \bar{\theta} \leq \theta_1$.

The above analysis applies to steady compressible fluids by replacing ρu for u and ρv for v . The assumption must be made that the density field ρ is continuous in $\partial U \partial \partial$ and that $\rho \neq 0$ or ∞ .

The above theorem, for incompressible fluid, involves only "instantaneous" considerations. The assumption that the flow field is steady is not critical to the argument. By assuming that (i) - (x) holds uniformly in time, the results can be applied to the non-steady case.

The assumption that the motion is two-dimensional is made out of convenience; nevertheless, the analysis has not been extended to the three-dimensional case.

VIII. To show that the forces exerted by fluids undergoing Stokes flow are unbounded in the neighborhood of the common line.

Section 1. Introduction

A moving common line formed by a fluid-fluid interface intersecting a planar rigid wall will be considered. It is assumed that the motion is two-dimensional and steady in a coordinate system moving with the common line. Only one of the fluids will be investigated at the moment, and this fluid will be considered to be Newtonian and incompressible and satisfy the no-slip boundary condition on the wall.

The aim of this chapter is the study of the force exerted by the fluid on the bounding surface. It will be shown that if the fluid in a neighborhood of the common line undergoes Stokes flow (i.e. very slow flow), then the force exerted by the fluid on the wall is unbounded.

The analysis, due to its length, is split up into nine sections. In section 2 it is shown that the biharmonic equation can be considered as a first order approximation to the Navier-Stokes equation. The existence and behavior of solutions to the biharmonic equation with appropriate boundary conditions are sought in a domain \mathcal{D} . In section 3 it is shown that the stream function can be divided into two parts, ψ_1 and \mathcal{U} , and that the first of these parts leads to the unbounded force. The purpose of the rest of this chapter is to establish that the remaining part \mathcal{U} of the stream function must lead

to a bounded force and hence the singular part due to Ψ_1 can never be cancelled by the addition of the second part. At the end of section 3 it is shown that if a certain function, $\phi(s)$, which is associated with \mathcal{U} , can be shown to be continuous, then this implies that the force exerted by \mathcal{U} must be bounded.

The big difficulty in this analysis comes from the fact that the domain \mathcal{D} has a corner. The corner occurs at the common line. All the methods written up in popular text books, such as Muskhelishvili (1954), Mikhlín (1946), and Sokolnikoff (1956), require the curvature of the boundary to be at least continuous. Hence the domain can have no corner. Suggestions are made in the discussions of these methods, to approximate the contour in the neighborhood of the corner by an infinite sequence of contours, each possessing continuous curvature and whose limit is the contour with the corner. However, these "suggestions" are not accompanied by a proof (except for some special cases) that the solutions to such an infinite sequence of problems will converge to a solution of the desired problem. It is for this reason that another technique has to be used.

In a brief note at the end of a section in Muskhelishvili (1954; p.418) reference is made to a series of papers by a Georgian (the place in the U.S.S.R.), Magnaradze (1937, 1938a, 1938b), who considers the biharmonic equation for a domain with corners. He shows that the biharmonic problem can be reduced

to an integral equation having an integral operator which is not completely continuous. He then follows, very closely, the work of Radon (1919). Radon had to solve a similar integral equation which occurs in the two-dimensional potential problem when the domain possesses a corner in its boundary. Radon shows how the integral operator can be divided up into two parts; the first part is completely continuous; in fact, it has a bounded continuous kernel, and the second part possesses a norm less than one. This enables Radon to manipulate the integral equation so as to put it in a form for which the Fredholm Alternative can be used to determine the conditions under which solutions exist. The equations of Radon and Magnaradze differ in two ways. Firstly, Magnaradze has a system of two simultaneous integral equations whereas Radon has only one. Secondly, Magnaradze has a slightly different kernel.

Magnaradze's analysis is obscure on two points. The first is that it is not quite clear how he defines the norm of his integral operator and it is even less clear that he establishes that the norm of the second part of his operator is less than one. The second is that he must verify that the solutions to his integral equations are Hölder continuous, and this is not done.

It is for these reasons, and the fact that the article of Magnaradze (1938b) which gives all the details, is written

in Russian in a journal that is not easily accessible, that the rest of this chapter is devoted to presenting a complete form of the analysis.

In section 4 a complete derivation is given of the two simultaneous integral equations which are equivalent to the biharmonic problem. The derivation follows very closely the case of a smooth contour. For this reason the derivation in Miklin (1957; pp.243-245) is closely paralleled with the differences being pointed out.

In section 5 the integral operator is decomposed into two points. In section 6, sufficient conditions are established under which the norm of the singular part is less than one. Magnaradze's analysis is criticized here. In section 7 the integral equation is manipulated. An explicit expression for the kernel of the new integral operator is derived. In section 8, it is shown that this kernel is Hölder continuous. In section 9, it is shown that any solution to this integral equation must also be Hölder continuous. None of this has been done by Magnaradze. In section 10 the Fredholm alternative is then used to establish necessary and sufficient conditions for which a solution exists to the integral equation. It also can be shown that the solution to this integral equation indeed gives rise to a solution to the biharmonic equation. This part of the analysis follows almost word for word from Miklin (1957, pp.245-249). The fact that the contour possesses a corner

leaves the analysis almost unchanged for this part. Consequently, the details will not be included.

Section 2. Derivation of the Governing Mathematical System.

A coordinate system is used in which the common line lies at the origin and the bounding surface coincides with the line $Y = 0$ (see figure (8.1)).

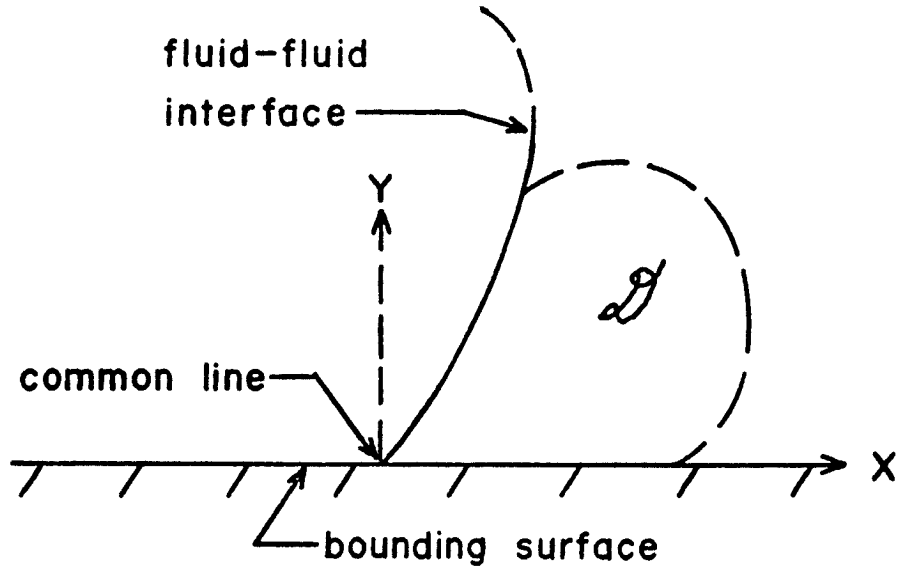


Fig. (8.1) The fluid lying within domain D is studied. The dashed part of ∂D which joins the fluid-fluid interface with $Y=0$ is arbitrary but fixed.

It is assumed that the curvature of ∂D exists and is Hölder continuous* everywhere except at the common line where

* A complex function $f(z)$ satisfies the Hölder condition for $z \in \partial D$ means (Muskhelishvili, 1963; p.260) that there exists two positive constants A and μ with $0 < \mu \leq 1$ such that for any two points on ∂D , say z_1 and z_2 , the following inequality holds true:

$$|f(z_1) - f(z_2)| < A |z_1 - z_2|^\mu.$$

only the left-handed and right-handed curvatures exist.

The tangential component of the force, denoted by $F(x_1)$, exerted by the Newtonian fluid on the portion of the bounding surface between $X=0$ and $X=x_1 > 0$ is given by:

$$F(x_1) = \mu \int_0^{x_1} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \Big|_{y=0} dx \quad (8.1)$$

where μ is a physical constant of the fluid called the absolute viscosity, and u and v are the cartesian components of the velocity field parallel to the X and Y axis respectively.

In order to be able to draw conclusions concerning the force acting on the boundary it is necessary to have some knowledge of the behavior of the velocity field. It can be shown, as a direct consequence of conservation of linear momentum, that the velocity field for an incompressible Newtonian fluid must satisfy a non-linear partial differential equation called the Navier-Stokes equation:

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right) = -\nabla p + \mu \nabla^2 \underline{u} \quad (8.2a)$$

and the conservation of mass equation: $\nabla \cdot \underline{u} = 0 \quad (8.2b)$

where P is the pressure, ρ is the density, $\underline{u} = (u, v)$ and t is the time. Due to its nonlinearity in the velocity, eqn (8.2) is difficult to solve; "exact" solutions are known only for a few particular situations. One technique of obtaining approximate solutions to eqn (8.2), for slow flows, is to scale the variables as follows:

$$\begin{aligned}\hat{\underline{u}} &\equiv \underline{u} (\mathcal{U}_0)^{-1} \\ \hat{P} &\equiv P \left(\frac{L}{\mathcal{U}_0 \mu} \right) \\ \hat{t} &\equiv t \frac{\mathcal{U}_0}{L} \\ (\hat{x}, \hat{y}) &\equiv (x, y) L^{-1}\end{aligned}$$

where \mathcal{U}_0 is the speed of the rigid wall (equal to minus the common line speed in laboratory coordinates) and L is the maximum distance between any two points on $\partial\mathcal{D}$. Eqn (8.2) then has the form

$$\begin{aligned}R \left\{ \frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + (\hat{\underline{u}} \cdot \hat{\nabla}) \hat{\underline{u}} \right\} &= -\hat{\nabla} \hat{P} + \hat{\nabla}^2 \hat{\underline{u}} \\ \hat{\nabla} \cdot \hat{\underline{u}} &= 0\end{aligned}$$

where the Reynolds number R is given by $R = \frac{\mathcal{U}_0 L \rho}{\mu}$

The leading term of an asymptotic expansion valid for $R \rightarrow 0$ satisfies the Stokes equations

$$0 = -\hat{\nabla} \hat{P}_0 + \hat{\nabla}^2 \hat{\underline{u}}_0 \quad (8.3a)$$

$$\hat{\nabla} \cdot \hat{u}_0 = 0 \quad (8.3b)$$

For infinite domains, this type of expansion could be non-uniformly valid (Van Dyke, 1964) but we are interested in only a small neighborhood of the common line so only a finite domain shall be considered. For two-dimensional flow, a stream function $\bar{\Psi}$ can be defined by

$$u_0 = \left(-\frac{\partial \bar{\Psi}}{\partial Y}, \frac{\partial \bar{\Psi}}{\partial X} \right)$$

In terms of $\bar{\Psi}$, it can easily be shown that eqns (8.3) reduce to

$$\nabla^4 \bar{\Psi} = 0 \quad (8.4)$$

(This equation and the rest of this section will be in terms of the dimensional variables.) Eqn (8.4) is known as Stokes equation, and a motion whose stream function $\bar{\Psi}$ obeys this equation is referred to as Stokes flow.

It will eventually be shown that if the stream function $\bar{\Psi}$ satisfies eqn (8.4), subject to appropriate boundary conditions on $\partial \mathcal{D}$ specified below, then the function F defined in eqn (8.1) is unbounded.

The boundary conditions on $\bar{\Psi}$ which are expressed in terms of u_0 are as follows:

- (i) $u_0 \cdot \gamma$ on $\partial \mathcal{D}$ is a continuous function of

arc length and satisfies the Hölder condition over the entire contour $\partial\mathcal{D}$, except at the origin (common line) where $\underline{\tau}$ is not well defined. $\underline{\tau}$ is the unit tangent vector to the contour $\partial\mathcal{D}$. At the origin it is assumed that $\underline{\tau}|_{s_0^+}$ and $\underline{\tau}|_{s_0^-}$ exist where S is a parameter which measures arc length and $S=s_0$ is at the origin. It is assumed that $\underline{u} \cdot \underline{\tau}|_{s_0^+} - \underline{u} \cdot \underline{\tau}|_{s_0^-} = \mathcal{V}_0 \neq 0$ The important point is that the above equation does not equate to zero.

(ii) $\underline{u} \cdot \underline{n}$ is a continuous function of S and satisfies the Hölder condition everywhere on $\partial\mathcal{D}$.

\underline{n} is the unit outward normal vector to the contour $\partial\mathcal{D}$. In addition $\underline{u} \cdot \underline{n} \equiv 0$ for all $S \in \partial\mathcal{D}_1$ where $\partial\mathcal{D}_1 \subset \partial\mathcal{D}$ and s_0 is an interior point of $\partial\mathcal{D}_1$; refer to figure (8.2).

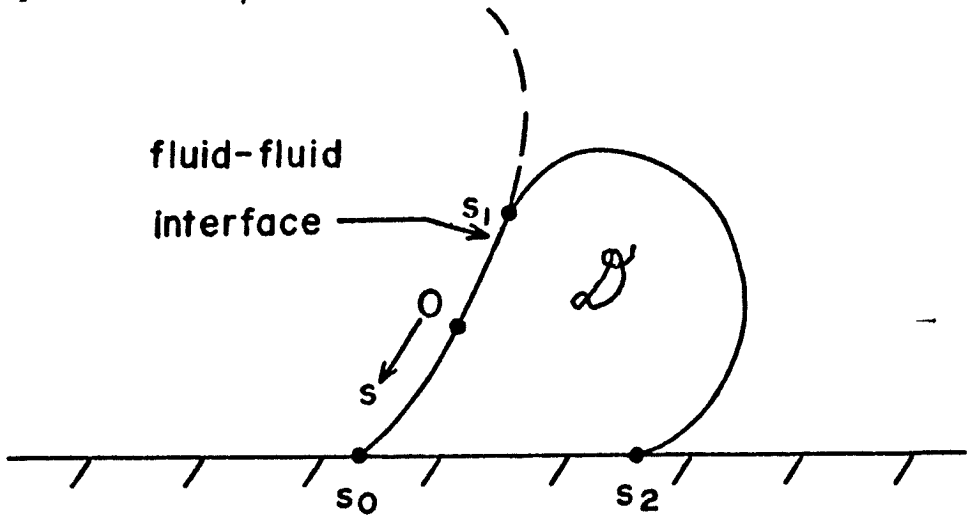


Fig. (8.2) $\partial\mathcal{D}_1$ is the portion of $\partial\mathcal{D}$ consisting of σ : $s_1 \leq \sigma \leq s_2$.

Section 3. Decomposition of Ψ

The stream function Ψ is divided into two parts

ψ_1 and \mathcal{U} :

$$\Psi = \psi_1 + \mathcal{U}$$

with the requirement that:

$$\nabla^4 \psi_1 = 0 \quad \text{AND} \quad \nabla^4 \mathcal{U} = 0 \quad \text{in } \mathcal{D}.$$

The function ψ_1 satisfies the following boundary conditions:

$$(iii) \quad \underline{u}_1 \cdot \underline{n}_w = 0 \quad \text{on} \quad \partial W = \partial W_1 \cup \partial W_2$$

where \underline{n}_w is the outward normal to ∂W and

$$\underline{u}_1 \equiv \left(-\frac{\partial \psi_1}{\partial y}, +\frac{\partial \psi_1}{\partial x} \right), \quad \text{refer to figure (8.3) for}$$

definition of the domain and boundary.

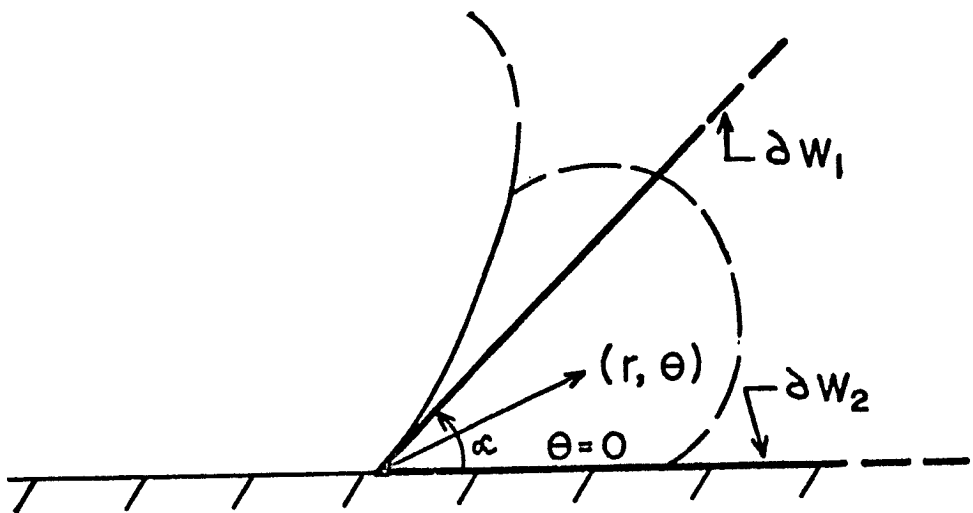


Fig. (8.3) The boundary ∂W forms a wedge at S_0 , whose sides are tangent to $\zeta(s_0^+)$ AND $\zeta(s_0^-)$.

(iv) $\underline{u}_1 \cdot \underline{\tau}_1 = 0$ on ∂W_1 where $\underline{\tau}_1$ is the unit tangent vector to ∂W_1

(v) $\underline{u}_1 \cdot \underline{\tau}_2 = U_0$ on ∂W_2 where $\underline{\tau}_2$ is the unit tangent vector to ∂W_2

A biharmonic function which satisfies (iii), (iv) and (v) is as follows:

$$\psi_1 = \frac{+U_0 r \{ \alpha(\theta - \alpha) \sin \theta + \theta \sin \alpha \sin(\alpha - \theta) \}}{\{ \alpha^2 - \sin^2 \alpha \}} \quad (8.5)$$

The function ψ_1 is defined over the whole domain (not only within the wedge). $\underline{u}_1 \cdot \underline{n}$ on $\partial \Omega$ is easily seen to satisfy the Hölder condition and

$$\lim_{\substack{s \rightarrow s_0 \\ s \in \partial \Omega}} \underline{u}_1 \cdot \underline{n} = 0.$$

Also, $\underline{u}_1 \cdot \underline{\tau}$ ($\underline{\tau}$ is tangent to $\partial \Omega$) satisfies the Hölder condition except at $s = s_0$ where:

$$\lim_{\substack{s \rightarrow s_0 \\ s < s_0 \\ s \in \partial \Omega}} \underline{u}_1 \cdot \underline{\tau} = 0 \quad \text{AND} \quad \lim_{\substack{s \rightarrow s_0 \\ s > s_0 \\ s \in \partial \Omega}} \underline{u}_1 \cdot \underline{\tau} = U_0.$$

The biharmonic function ψ is taken to have those boundary conditions which force the sum $\Psi = \psi_1 + \psi$ to assume its correct boundary values. These boundary conditions are as follows:

(vi) $u_{2 \cdot n} \equiv u \cdot n - u_{1 \cdot n}$ where $u_2 \equiv \left(-\frac{\partial U}{\partial Y}, \frac{\partial U}{\partial X}\right)$

As a consequence of the above mentioned properties of ψ_1 , it is known that $u_{1 \cdot n}$ is continuous and satisfies the Hölder condition everywhere on $\partial \mathcal{D}$.

(vii) $u_{2 \cdot \tau} \equiv u \cdot \tau - u_{1 \cdot \tau}$ For the same reasons as above, $u_{2 \cdot \tau}$ is continuous and satisfies the Hölder condition everywhere on $\partial \mathcal{D}$.

The horizontal component F of the force exerted by the fluid on the bounding surface at $Y=0$ and on the segment $0 < X < X_1$ is given by eqn (8.1) and in terms of the functions ψ_1 and U is given by:

$$F(x_1) = \mu \int_0^{x_1} \left(\frac{\partial^2 \psi_1}{\partial X^2} - \frac{\partial^2 \psi_1}{\partial Y^2} \right) \Big|_{Y=0} dx + \mu \int_0^{x_1} \left(\frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} \right) \Big|_{Y=0} dx \quad (8.6)$$

If the solution (8.5) is substituted into the first term of eqn (8.6),

$$\begin{aligned} \mu \int_0^{x_1} \left(\frac{\partial^2 \psi_1}{\partial X^2} - \frac{\partial^2 \psi_1}{\partial Y^2} \right) \Big|_{Y=0} dx &= \\ &= \frac{-\mu U_0 (2\alpha - \sin 2\alpha)}{(d^2 - \sin^2 \alpha)} \int_0^{x_1} \frac{1}{r} dr \end{aligned}$$

is obtained.

The above integral is unbounded for any $\chi_1 \neq 0$. Hence the part of the stream function given by ψ_1 gives rise to an unbounded force. It is now shown that the second integral in eqn (8.6) is bounded and so it can never cancel the contribution of the first integral. Therefore, we conclude that $F(\chi_1)$ is unbounded for any χ_1 .

In order to demonstrate that the second integral in eqn (8.6) is bounded we must first use a theorem due to Muskhelishvili (1963; p.110). Any solution to the biharmonic equation, i.e.

$$\nabla^4 U = 0,$$

can be represented in the form:

$$U = \text{Re} \left\{ \bar{z} \varphi(z) + \psi(z) \right\} \quad (8.7)$$

where $\varphi(z)$ and $\psi(z)$ are analytic functions of a complex variable z , $z = x + iy$, $\bar{z} = x - iy$, and Re denotes the "real part" of the complex quantity within the brackets. In addition, any function U which is represented in this form is a solution to the biharmonic equation. For the functions $\varphi(z)$ and $\psi(z)$ to be analytic implies that their real and imaginary parts must be harmonic, i.e. if

$P(x,y)$, $Q(x,y)$ and $t(x,y)$ are defined to be real valued functions such that:

$$\varphi(z) = P(x, y) + i Q(x, y) \quad \text{AND} \quad t(x, y) = R e^{-\psi(z)} \quad (8.8)$$

then the functions $\varphi(z)$ and $\psi(z)$ are analytic. This implies that the Cauchy-Riemann conditions must be satisfied:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad \text{AND} \quad \frac{\partial P}{\partial y} = - \frac{\partial Q}{\partial x}$$

and so

$$\nabla^2 P = 0, \quad \nabla^2 Q = 0, \quad \nabla^2 t = 0$$

If eqn (8.8) is substituted into eqn (8.7), then

$$U = x P(x, y) + y Q(x, y) + t(x, y)$$

If it is assumed that the derivatives of U on ∂D are the limits of the derivatives in D as they approach ∂D (which will be the case), then using the boundary conditions imposed on U it follows that:

$$\int_0^{x_1} \left[\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} \right] \Big|_{y=0} dx = \int_0^{x_1} \left[- \frac{\partial^2 U}{\partial y^2} \right] \Big|_{y=0} dx \quad (8.9)$$

since $v = \frac{\partial U}{\partial x} \equiv 0$ on $y=0$ and hence $\frac{\partial^2 U}{\partial x^2} \equiv 0$

on $y=0$. This also means that:

$$\nabla^2 U \Big|_{Y=0} = \frac{\partial^2 U}{\partial Y^2} \Big|_{Y=0} \quad (8.10)$$

$\nabla^2 U$ can be evaluated in general to yield that

$$\nabla^2 U = 2 \left\{ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial Y} \right\} = 4 \frac{\partial P}{\partial x} = 4 \frac{\partial Q}{\partial Y} \quad (8.11)$$

If eqn (8.11) and eqn (8.10) are substituted into eqn (8.9), then

$$\begin{aligned} - \int_0^{x_1} \left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial Y^2} \right) \Big|_{Y=0} dx &= 4 \int_0^{x_1} \frac{\partial P}{\partial x} \Big|_{Y=0} dx \\ &= P(x_1, 0) - P(0, 0) \end{aligned} \quad (8.12)$$

The aim of the rest of this chapter is to show that U , subject to the boundary condition in (vi) and (vii), gives rise to a function $\Phi(z)$ which is continuous on the entire boundary $\partial \mathcal{D}$. In particular, it is continuous at the common line (the corner). Upon demonstrating this it can be concluded that:

$$| P(x_1, 0) - P(0, 0) | \quad \text{is bounded,}$$

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and therefore that $\int_0^{x_1} \left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \right] dx$ is bounded

and finally that:

$$F(x_1) \quad \underline{\text{is unbounded.}}$$

Parenthetically, it is interesting to note that the fluid cannot exert any concentrated tangential force on the boundary. A concentrated tangential force is defined as:

$$\lim_{x_1 \rightarrow 0} F(x_1) \neq 0$$

This is a bounded force produced by an unbounded stress tensor.

For a concentrated tangential force to exist on the bounding surface $P(x, 0)$ must be discontinuous (refer to eqn (8.12)).

However, this cannot take place if it is demonstrated that

$$P(x, y) \text{ is continuous function for } x, y \in \partial \mathcal{D}.$$

Section 4. Derivation of integral equation for $\phi(s)$

Instead of solving for U directly, we solve for the two analytic functions $\phi(z)$ and $\psi(z)$. The first step is to derive an integral equation for the variable $\phi(s)$ which is the value of the function ϕ on the boundary ∂D ; the parameter s denotes a measure of the arc length of ∂D . (The notation is a bit sloppy. Actually $z = z(s)$ gives the correspondence between a point on ∂D at a particular s and the position of that point in terms of its x and y coordinates, i.e. $z = x(s) + iy(s)$. Thus, $\phi(z) = \phi(z(s)) = \hat{\phi}(s)$.) The derivation of the integral equation for the case when a corner is located on ∂D is due to Magnaradze (1938b). The derivation that follows follows almost word for word the one appearing in Mikhlin (1957; p.243-245) which assumes no corners. The parts of the derivation which differ due to the presence of a corner are pointed out.

Recall the equation relating U , $\phi(z)$ and $\chi(z)$:

$$U = \operatorname{Re} \left\{ \bar{z} \phi(z) + \chi(z) \right\} \quad (8.7)$$

If eqn (8.7) is differentiated with respect to x and y separately, then $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ are given in terms

of $\phi(z)$ and $\psi(z) \equiv \frac{d\chi(z)}{dz}$ (not to be confused with the stream function Ψ):

$$\frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = \varphi(z) + z \overline{\varphi'(z)} + \overline{\psi(z)} .$$

The quantity $f(z)$ is defined by:

$$f(z) \equiv \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} \tag{8.13a}$$

This function is called the complex velocity. \mathcal{D}' denotes the region (infinite) complimentary to $\partial \cup \partial \mathcal{D}$. z' is any arbitrary point in \mathcal{D}' . The complex conjugate of the above equation with z replaced by ζ gives

$$\overline{\varphi(\zeta)} + \bar{\zeta} \varphi'(\zeta) + \psi(\zeta) = \overline{f(\zeta)} \tag{8.13b}$$

where ζ is the complex variable z restricted to $\partial \mathcal{D}$, i.e. $\zeta \in \partial \mathcal{D}$. Eqn (8.13b) is multiplied by $\frac{1}{2\pi i} \cdot \frac{1}{\zeta - z'}$, and integrated over the contour $\partial \mathcal{D}$; this gives:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{\overline{\varphi(\zeta)}}{\zeta - z'} d\zeta + \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{\bar{\zeta} \varphi'(\zeta)}{\zeta - z'} d\zeta + \\ + \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{\psi(\zeta)}{\zeta - z'} d\zeta = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{\overline{f(\zeta)}}{\zeta - z'} d\zeta . \end{aligned} \tag{8.14}$$

As a consequence of $\phi(z)$, $\phi'(z)$ and $\psi(z)$ being analytic in \mathcal{D} and the fact that the point z' lies outside \mathcal{D} , the following relationships hold:

$$(a) \quad \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\phi(\xi)}{\xi - z'} d\xi \equiv 0$$

$$(b) \quad \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\phi'(\xi)}{\xi - z'} d\xi \equiv 0$$

$$(c) \quad \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\psi(\xi)}{\xi - z'} d\xi \equiv 0$$

If eqn (c) is substituted into eqn (8.14), then:

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\overline{\phi(\xi)}}{\xi - z'} d\xi + \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\bar{\xi} \phi'(\xi)}{\xi - z'} d\xi = A(z') \quad (8.15)$$

where

$$A(z') \equiv \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{\overline{f(\xi)}}{\xi - z'} d\xi.$$

If all quantities in eqn (a) are replaced by their complex conjugates, if eqn (b) is multiplied by $-\bar{z}'$ and both of these are added to eqn (8.15), then

$$\frac{1}{2\pi i} \int_{\partial D} \overline{\varphi(\zeta)} \left\{ \frac{d\zeta}{\zeta - z'} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}'} \right\} + \frac{1}{2\pi i} \int_{\partial D} \left\{ \frac{\bar{\zeta} - \bar{z}'}{\zeta - z'} \right\} \varphi'(\zeta) d\zeta = \quad (8.16)$$

$$= A(z')$$

If the integral involving $\varphi'(\zeta)$, on the left hand side of eqn (8.16) is integrated by parts and it is assumed that $\varphi(\zeta)$ is a continuous function, it then follows that

$$\frac{1}{2\pi i} \int_{\partial D} \overline{\varphi(\zeta)} \left\{ \frac{d\zeta}{\zeta - z'} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}'} \right\} - \frac{1}{2\pi i} \int_{\partial D} \varphi(\zeta) d \left\{ \frac{\bar{\zeta} - \bar{z}'}{\zeta - z'} \right\} = \quad (8.17)$$

$$= A(z')$$

The analysis, so far, has been taken directly from Mikhlin (1957; p.243). The limit $\bar{z}' \rightarrow t$ of eqn (8.17) is now taken where t is a point on the contour ∂D . The following formula due to Plemelj (Muskhelishvili, 1946; p.429) is used:

$$\lim_{z' \rightarrow t} \frac{1}{2\pi i} \int_{\partial D} \frac{g(\zeta)}{\zeta - z'} d\zeta = \frac{-\alpha}{2\pi} g(t) + \frac{1}{2\pi i} \int_{\partial D} \frac{g(\zeta)}{\zeta - t} d\zeta -$$

where $g(\zeta)$ is assumed to satisfy the Hölder condition, the integral on the right is to be interpreted as the principal

value, and α is the inner angle of $\partial\mathcal{D}$ at t ; refer to figure (8.3). This formula is even valid for the

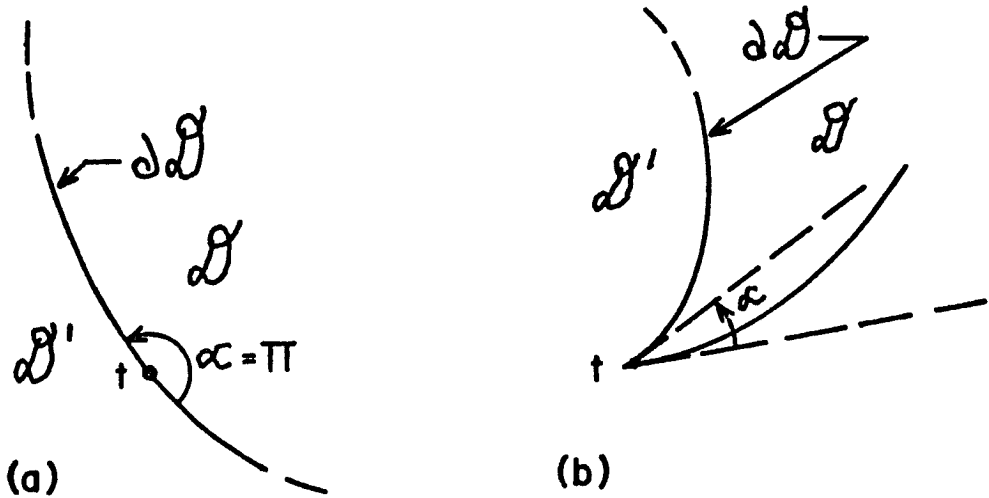


Fig. (8.3) (a) The point t lies on a smooth portion of $\partial\mathcal{D}$, $\alpha = \pi$.
 (b) The point t lies at a corner. The angle α is that formed between the right and left handed tangents at t , and lying in \mathcal{D} .

special case when the corner is a cusp, i.e. $\alpha = 0$. The above mentioned limit of eqn (8.17) gives:

$$\begin{aligned}
 & -\frac{\alpha}{\pi} \overline{\varphi(t)} + \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \varphi(\xi) d \ln \left\{ \frac{\xi - t}{\bar{\xi} - \bar{t}} \right\} + \\
 & - \frac{1}{2\pi i} \lim_{z' \rightarrow t} \int_{\partial\mathcal{D}} \varphi(\xi) d \left\{ \frac{\bar{\xi} - \bar{z}'}{\xi - z'} \right\} = A(t)
 \end{aligned} \tag{8.18}$$

where the first integral on the left hand side is evaluated

for its principal value and $A(t) \equiv \lim_{z' \rightarrow t} A(z')$,

$$A(t) = \frac{-\alpha}{2\pi} \overline{f(t)} + \frac{1}{2\pi i} \int_{\partial D} \frac{\overline{f(\zeta)}}{\zeta - t} d\zeta.$$

It can be shown that if $\varphi(\zeta)$ is continuous, then

$$\begin{aligned} \lim_{z' \rightarrow t} \int_{\partial D} \varphi(\zeta) d\zeta (e^{-2i\theta(\zeta, z')}) &= \int_{\theta(t, t^+)}^{\theta(t, t^-)} \varphi(\zeta) d\zeta e^{-2i\theta(t, \zeta)} \\ &+ \varphi(t) \left\{ e^{-2i\theta(t, t^-)} - e^{-2i\theta(t, t^+)} \right\} \end{aligned}$$

where $\theta(\zeta, z')$ is the angle between the vector $z' - \zeta$ and the Ox axis, and the symbol $d\zeta$ () means that the variable of integration is ζ . The change of variables

$$\zeta - t = r e^{i\theta(t, \zeta)}$$

in eqn (8.18) is made and the above equation is substituted

into eqn (8.18). The result is

-

$$\begin{aligned}
 & -\frac{\alpha}{\pi} \bar{\varphi}(t) + \frac{\varphi(t)}{2\pi i} \left\{ e^{-2i(\alpha+\beta)} - e^{-2i\beta} \right\} + \\
 & + \frac{1}{\pi} \int_{\beta}^{\alpha+\beta} \overline{\varphi(\zeta)} d\zeta \theta(t, \zeta) + \frac{1}{\pi} \int_{\beta}^{\alpha+\beta} \varphi(\zeta) e^{-2i\theta(t, \zeta)} d\zeta \theta(t, \zeta) = \\
 & = A(t) \tag{8.19}
 \end{aligned}$$

where $\beta \equiv \theta(t, t^+)$. If t is at a smooth point on the boundary then $\alpha = \pi$ and eqn (8.19) becomes

$$\begin{aligned}
 & -\overline{\varphi(t)} + \frac{1}{\pi} \int_{\beta}^{\beta+\pi} \left\{ \overline{\varphi(\zeta)} + \varphi(\zeta) e^{-2i\theta(t, \zeta)} \right\} d\zeta \theta(t, \zeta) = \\
 & = A(t)
 \end{aligned}$$

which is identical to the equation in Mikhlin*. Equation (8.19) can be written in a more compact form:

* Mikhlin's eqn (7) on page 245 should have $-A(t)$ for $A(t)$. This can be cross-checked with eqn (98.91) in Muskhelishvili, 1963; p.411.

$$-\overline{\varphi(t)} + \frac{1}{\pi} \int_{\beta}^{\beta+\pi} \left\{ \overline{\varphi(\xi)} + \varphi(\xi) e^{-2i\theta(t,\xi)} \right\} d\xi \theta(t,\xi) = A(t) \quad (8.20)$$

which is interpreted as a Stieltjes integral when t is at a corner. Again, recognition of this is due to Magnaradze (1938b).

Eqn (8.20) can be put into a slightly different form. The function $\Theta(s,\sigma)$ is defined by the following:

$$\Theta(s,\sigma) \pmod{2\pi} = \theta(s,\sigma) + \pi \mu(s-\sigma)$$

where s and σ are the solutions to $t = z(s)$ and $\xi = z(\sigma)$ respectively.

The function $\mu(x)$ is the step function defined by:

$$\mu(x) = \begin{cases} 0 & \text{FOR } x < 0 \\ 1 & \text{FOR } x \geq 0 \end{cases}$$

The notation is, again, ambiguous since $\theta(t,\xi) = \theta(s,\sigma)$. In addition, the quantity $\theta(s,s)$ is defined as:

$$\theta(s,s) \equiv \theta(s,s^+)$$

Using the above notation, eqn (8.20) can be rewritten as:

$$-\overline{\varphi(s)} + \frac{1}{\pi} \int_0^1 \left\{ \overline{\varphi(\sigma)} + \varphi(\sigma) e^{-2i\Theta(s,\sigma)} \right\} d\sigma \Theta(s,\sigma) = A(s) \quad (8.21)$$

where $s=0$ and $s=l$ are the same points on ∂D , and ∂D has a length l . So the integral equation can be written either in terms of $\theta(s, \sigma)$ or $\Theta(s, \sigma)$ as long as the correct limits of integration are used.

Eqn (8.21) can also be written as two simultaneous linear integral equations in terms of the real variables $P(s)$ and $q(s)$ (Magnaradze, 1938b) where

$$\varphi(z(s)) = P(s) + i q(s),$$

and $\xi = z(s)$

Eqn (8.21) then becomes:

$$P(s) - \frac{1}{\pi} \int_{\partial D} \left\{ P(\sigma) [1 + \cos 2\theta(s, \sigma)] + q(\sigma) \sin 2\theta(s, \sigma) \right\} d_\sigma \theta(s, \sigma) = B_1(s)$$

$$q(s) - \frac{1}{\pi} \int_{\partial D} \left\{ P(\sigma) \sin 2\theta(s, \sigma) + q(\sigma) [1 - \cos 2\theta(s, \sigma)] \right\} d_\sigma \theta(s, \sigma) = B_2(s)$$

(8.22)

where $A(s) = -B_1(s) + i B_2(s)$. The system (8.22) can be written in the form of a single equation provided the following

change in variables and definitions of terms are made:

$$\Phi(s) \equiv P(s) \quad \text{FOR} \quad 0 \leq s < l$$

$$\Phi(s) \equiv q(s-l) \quad \text{FOR} \quad l \leq s < 2l$$

$$D(s) \equiv B_1(s) \quad \text{FOR} \quad 0 \leq s < l$$

$$D(s) \equiv B_2(s) \quad \text{FOR} \quad l \leq s < 2l$$

where l is the length of contour γ , and

$$\gamma(s, \sigma) \equiv \begin{cases} \theta(s, \sigma) + \frac{1}{2} \sin 2\theta(s, \sigma) & \text{FOR } 0 \leq s < l; \\ & 0 \leq \sigma < l \\ -\frac{1}{2} \cos 2\theta(s, \sigma) & \text{FOR } 0 \leq s < l; \\ & l \leq \sigma < 2l \\ -\frac{1}{2} \cos 2\theta(s, \sigma) & \text{FOR } l \leq s < 2l; \\ & 0 \leq \sigma < l \\ \theta(s, \sigma) - \frac{1}{2} \sin 2\theta(s, \sigma) & \text{FOR } l \leq s < 2l; \\ & l \leq \sigma < 2l \end{cases}$$

where $\theta(s, \sigma) \equiv \theta(s+l, \sigma) \equiv \theta(s, \sigma+l) \equiv \theta(s+l, \sigma+l)$.

System (22) becomes:

$$\Phi(s) - \frac{1}{\pi} \int_{\mathcal{C}} \Phi(\sigma) d_\sigma \gamma(s, \sigma) = D(s) \quad (8.23)$$

where the scalars s and σ now vary over the extended interval $\mathcal{C} : [0, 2l]$.

Section 5. Method of approach

A linear operator T is defined as follows:

$$T\Phi(s) \equiv \frac{1}{\pi} \int_{\mathcal{L}} \Phi(\sigma) d_{\sigma} \chi(s, \sigma).$$

Eqn (8.23) can then be rewritten as:

$$\Phi(s) - T\Phi(s) = D(s). \quad (8.24)$$

For the case when \mathcal{D} possesses a smooth boundary, it can be shown (Muskhelishvili, 1963; p.412) that eqn (8.24) is a Fredholm equation, that is, the kernel of T is \mathcal{L}^2 and the Fredholm Alternative can be used to establish the conditions under which a solution exists. However, this is not the case here since the boundary $\partial\mathcal{D}$ possesses a corner. It has been shown in section 4 that the function $\Theta(s_0, \sigma)$ has a jump of magnitude $\pi^{-\alpha}$ at $\sigma = s_0$ (where $s = s_0$ is the location of the corner). This means that $\chi(s_0, \sigma)$ is also discontinuous at $\sigma = s_0$ and hence T is not a completely continuous operator (Riesz and Nagy, 1955; p.221). Hence the Fredholm Alternative cannot be directly applied. Furthermore, the norm $\|T\|$ of T , defined by:

$$\|T\| \equiv \sup_{0 \leq s \leq 2\ell} \int_{\mathcal{L}} |d_{\sigma} \chi(s, \sigma)|, \quad (8.25)$$

is not less than one and so the inverse of operator $I-T$ cannot be represented as a convergent Neumann series.

The existence and behavior of solutions of eqn (8.24) can be shown, as suggested by Magnaradze (1938b), by using a technique due to Radon (1919a). A discussion of this method in English can be found in Riesz and Nagy (1955; p.219). Radon (1919) applies this technique to an integral equation obtained for the two-dimensional potential problem; the integral equation is almost the same as eqn (8.24). The technique consists of writing T as the sum of two linear operators T_1 and V ; i.e.

$$T = T_1 + V$$

The operator T_1 contains the "singular" part of T and furthermore has the property that $\|T_1\| < 1$. This means the inverse of the operator $I-T_1$ exists and can be expressed as a convergent Neumann series; i.e.

$$(I - T_1)^{-1} \Phi(s) = \sum_{n=0}^{\infty} T_1^n \Phi(s)$$

where

$$T^0 \Phi(s) \equiv \Phi(s)$$

and

$$T^n \Phi(s) \equiv [T(T^{n-1} \Phi)](s) \quad \text{FOR } n > 0$$

The operator V is completely continuous. Then one applies $(I - T_1)^{-1}$ to eqn (8.24) and obtains:

$$\Phi(s) - (I - T_1)^{-1} V \Phi(s) = (I - T_1)^{-1} D(s) \quad (8.26)$$

where the operator $(I - T_1)^{-1} V$ is completely continuous (Riesz and Nagy, 1955; p.178). Thus the Fredholm Alternative can be used to discuss the solution of eqn (8.26).

The above technique applied to eqn (8.24) requires the definition of the linear operators T_1 and V :

$$T_1 \Phi(s) \equiv \frac{1}{\pi} \int_{\mathcal{C}_{s_0, \epsilon}} \Phi(\sigma) d\sigma \zeta(s, \sigma)$$

where $\mathcal{C}_{s_0, \epsilon}$ consists of σ in the intervals $[s_0 - \epsilon, s_0 + \epsilon]$ and $[s_0 + l - \epsilon, s_0 + l + \epsilon]$ for any small positive number ϵ .

$$V \Phi(s) \equiv \frac{1}{\pi} \int_{\mathcal{C} - \mathcal{C}_{s_0, \epsilon}} \Phi(\sigma) d\sigma \zeta(s, \sigma)$$

Both $T_1 \Phi(s)$ and $V \Phi(s)$ are defined for any $s \in \mathcal{C}$.

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Section 6. Conditions for $\|T_i\| < 1$

Sufficient conditions are now derived such that $\|T_i\| < 1$. It is shown that if $\alpha > \alpha_0$ where $60^\circ < \alpha_0 < 90^\circ$ (α is the inner angle at the corner), then $\|T_i\| < 1$.

It has been assumed that the curvature of ∂D both exists and is Hölder continuous everywhere except at the corner where only the "one sided" curvatures exist. This implies that a small positive number δ can be chosen such that the segments of the contour lying in the intervals $[s_0 - \delta, s_0]$ and $[s_0, s_0 + \delta]$ lie as close as is wished to two straight lines emerging from s_0 , of length δ , and tangent to ∂D at $s = s_0$; refer to figure (8.4).

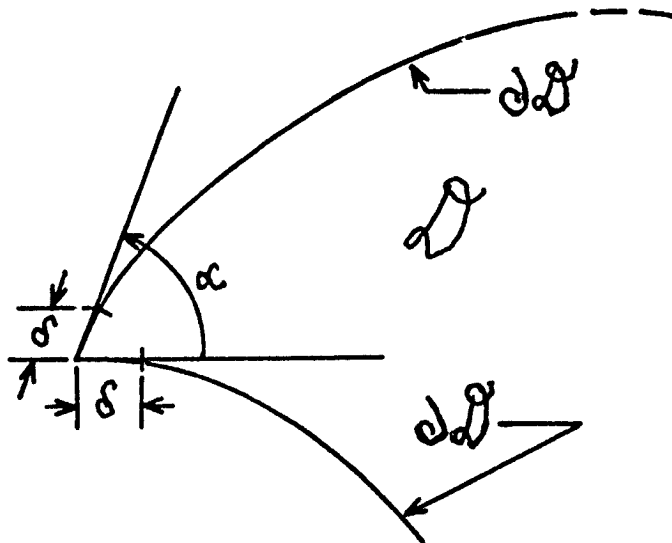


Fig. (8.4) Description of ∂D near $s = s_0$

The quantity δ is also chosen so small so that the segments

of the contour never cross the straight line. It is now shown for any fixed $\delta > 0$ that

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_{s_0, \epsilon}} |d_\sigma \zeta(s, \sigma)| = 0$$

uniformly in S for $s \in \Gamma - \Gamma_{s_0, \delta}$ where $\Gamma_{s_0, \delta}$ has an analogous definition as $\Gamma_{s_0, \epsilon}$. This explicitly means that for any $\gamma > 0$ there exists $\delta > 0$ such that

$$\int_{\Gamma_{s_0, \delta}} |d_\sigma \zeta(s, \sigma)| < \gamma \quad \text{FOR ANY } \epsilon < \delta$$

and for any $s \in \Gamma - \Gamma_{s_0, \delta}$. For $s \in (\Gamma - \Gamma_{s_0, \delta}) \cap [0, \ell)$ it follows that

$$\begin{aligned} \int_{\Gamma_{s_0, \delta}} |d_\sigma \zeta(s, \sigma)| &= \\ &= \frac{1}{\pi} \int_{\Gamma_{s_0, \delta}} \left\{ |1 + \epsilon_0 2\theta(s, \sigma)| + |\sin 2\theta(s, \sigma)| \right\} \\ &\quad \left| \frac{\partial \theta(s, \sigma)}{\partial \sigma} \right| d\sigma. \end{aligned}$$

For $s \in (\Gamma - \Gamma_{s_0, \delta}) \cap [\ell, 2\ell)$ it follows that

$$\int_{\rho_{s_0 \in}} \left| d_{\sigma} \gamma(s, \sigma) \right| = \frac{1}{\pi} \int_{\rho_{s_0 \in}} \left\{ \left| \sin 2\theta(s, \sigma) \right| + \left| 1 - \cos 2\theta(s, \sigma) \right| \right\}$$

$$\left| \frac{\partial \theta}{\partial \sigma}(s, \sigma) \right| d\sigma.$$

It can easily be shown that $\frac{\partial \theta(s, \sigma)}{\partial \sigma} = \frac{\cos(\underline{n}(\sigma), \vec{s}\sigma)}{r}$
 where $\underline{n}(\sigma)$ is the outward normal of ∂D at σ ,
 $\vec{s}\sigma$ is the position vector of the point in space at σ
 with respect to S , and $r = |\vec{s}\sigma|$.

Since

$$\left| 1 \pm \cos 2\theta(s, \sigma) \right| \leq 2 \quad \text{AND} \quad \left| \sin 2\theta(s, \sigma) \right| \leq 1,$$

$$\text{and} \quad \left| \cos(\underline{n}(\sigma), \vec{s}\sigma) \right| \leq 1,$$

it follows that

$$\int_{s_0 \in} \left| d_{\sigma} \gamma(s, \sigma) \right| \leq \int_{\rho_{s_0 \in}} \frac{3}{\pi r} d\sigma.$$

In addition, for any $S \in \mathcal{F} - \rho_{s_0 - \delta}$

$$\frac{1}{r} < \frac{1}{\delta - \epsilon} >$$

so that

$$\int_{\mathcal{C}_{s_0, \epsilon}} |d\sigma \chi(s, \sigma)| < \frac{b\epsilon}{\pi(\delta - \epsilon)}$$

ϵ is chosen to be

$$\epsilon \equiv \frac{\gamma\delta}{\left(\frac{b}{\pi} + \gamma\right)}$$

Hence, it has been shown that for any $\delta > 0$

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{C}_{s_0, \epsilon}} |d\sigma \chi(s, \sigma)| = 0 \quad \text{UNIFORMLY IN } S \text{ FOR } S \in \mathcal{C} - \mathcal{C}_{s_0, \delta}.$$

$$\int_{\mathcal{C}_{s_0, \epsilon}} |d\sigma \chi(s, \sigma)| \quad \text{for } S \in \mathcal{C}_{s_0, \delta} \quad \text{is}$$

now investigated but before this is done the geometry of $\mathcal{C}_{s_0, \delta}$ must be examined.

The quantities $\beta_1(\delta)$ and $\beta_2(\delta)$ are defined as follows:

$$\beta_1(\delta) \equiv \left| \theta(s_0 - \delta, (s_0 - \delta)^-) - \theta(s_0, s_0^-) \right|$$

and

$$\beta_2(\delta) \equiv \left| \theta(s_0 + \delta, (s_0 + \delta)^+) - \theta(s_0, s_0^+) \right|$$

Since the tangent to $\partial\mathcal{D}$ is continuous on both sides of the corner, it follows that

$$\lim_{\delta \rightarrow 0} \beta_1(\delta) = 0 \quad \text{AND} \quad \lim_{\delta \rightarrow 0} \beta_2(\delta) = 0.$$

As a consequence of the definition of the curvature $\mathcal{K}(s)$ of $\partial\mathcal{D}$,

$$\mathcal{K}(s) \equiv \left| \frac{\partial \theta(s, s^\pm)}{\partial s} \right|,$$

and that the one-sided curvatures at s_0 exist and are finite (i.e. the limits $\mathcal{K}(s_0^+)$ and $\mathcal{K}(s_0^-)$ of $\mathcal{K}(s)$ as $s \rightarrow s_0$ for $s > s_0$ and $s \rightarrow s_0$ for $s < s_0$ respectively exist) it follows that

$$\beta_1(\delta) \approx \mathcal{K}(s_0^-) \delta \quad \text{AND} \quad \beta_2(\delta) \approx \mathcal{K}(s_0^+) \delta$$

for sufficiently small $\delta > 0$.

Examination of $\theta(s, \sigma)$ for fixed s where $s \in \mathcal{C}_{\delta, \delta}$ and $s \neq s_0$ is made as σ varies from $s_0 - \epsilon$ to $s_0 + \epsilon$. For definiteness we consider $s_0 < s < s_0 + \delta$. As σ varies from $s_0 - \epsilon$ to s_0 , $\theta(s, \sigma)$ is either a monotonic function of σ or starts off monotonically, changes in value by a quantity whose magnitude is at most $\beta_1(\delta)$ and then proceeds monotonically to the quantity $\theta(s, s_0)$. As σ varies from s_0 to s^+ , $\theta(s, \sigma)$ varies monotonically. Finally, as σ varies from s^+ to $s_0 + \epsilon$, $\theta(s, \sigma)$

again varies monotonically. The sum of the magnitudes of these last two variations is at most $\beta_2(\delta)$. The function $\theta(s, \sigma)$ behaves in a similar manner if $s_0 - \delta < s < s_0$. It is also evident that if $s_0 < s < s_0 + \delta$, then

$$|\theta(s, s_0 - \epsilon) - \theta(s, s_0)| \leq \pi - \alpha + \beta_1(\delta)$$

and if $s_0 - \delta < s < s_0$ then

$$|\theta(s, s_0 + \epsilon) - \theta(s, s_0)| \leq \pi - \alpha + \beta_2(\delta).$$

In addition, we observe that if $\theta(s, \sigma)$ varies monotonically as σ varies from σ_0 to σ_1 , then

$$\int_{\sigma=\sigma_0}^{\sigma=\sigma_1} \left\{ \left| 1 \pm \cos 2\theta(s, \sigma) \right| + \left| \sin 2\theta(s, \sigma) \right| \right\} d\theta \leq 3 \left| \theta(s, \sigma_0) - \theta(s, \sigma_1) \right|$$

Using all the above results leads to the estimate

$$\|T_1\| \leq \max_{\theta(s_0, s_0)}^{\theta(s_0, s_0) + \pi} \frac{1}{\pi} \int \left\{ \left| 1 \pm \cos 2\theta(s, \sigma) \right| + \left| \sin 2\theta(s, \sigma) \right| \right\} d\theta + \beta(\delta) + \bar{\beta}(\epsilon),$$

WHERE $\lim_{\delta \rightarrow 0} \beta(\delta) = 0$ AND $\lim_{\epsilon \rightarrow 0} \bar{\beta}(\epsilon) = 0$.

Since the function $\theta(s, \sigma) \pm \frac{1}{2} \sin 2\theta(s, \sigma)$ is a monotonic function of $\theta(s, \sigma)$, we have

$$\int_{\theta(s_0, s_0^-)}^{\theta(s_0, s_0^+) + \pi} |1 \pm \cos 2\theta(s, \sigma)| d\theta = \pi - \alpha \pm \frac{1}{2} \sin 2\theta \left. \begin{array}{l} \theta = \theta(s_0, s_0^+) + \pi \\ \theta = \theta(s_0, s_0^-) \end{array} \right\} \cdot$$

It is noticed that the function $-\frac{1}{2} \cos 2\theta$ is not monotonic; hence the total variation of $-\frac{1}{2} \cos 2\theta$ is not the difference in the values at its end points. It is easily seen that there exists an angle α_0 where $60^\circ < \alpha_0 < 90^\circ$ such that if α is restricted to the interval $\alpha_0 < \alpha \leq \pi$, then

$$\|T_1\| < 1.$$

If the boundary has more than one corner, say N corners, located at s_1, s_2, \dots, s_N , then the operator T_1 is to be defined as:

$$T_1 \Phi(s) = \frac{1}{\pi} \int_{\rho_{s_1} \cup \rho_{s_2} \cup \dots \cup \rho_{s_N}} \Phi(\sigma) d\sigma \chi(s, \sigma)$$

When $s \in \rho_{s_i}$ then it can be shown that

$$\int_{\rho_{s_1} \cup \dots \cup \rho_{s_{i-1}} \cup \rho_{s_{i+1}} \cup \dots \cup \rho_{s_N}} |d\sigma \chi(s, \sigma)| = O(\epsilon)$$

and hence the problem reduces to that of one corner, and we get

$$\|T_1\| < 1 \quad \text{FOR} \quad \alpha_0 < \alpha \leq \pi .$$

In evaluating the norm of T_1 , Magnaradze (1938b) uses another approach which is not completely clear. He looks at the two separate cases when:

$$\bar{T}_1 \Phi(s) \equiv \frac{1}{\pi} \int_{s_0-\epsilon}^{s_0+\epsilon} \Phi(\sigma) d\sigma \left\{ \theta(s, \sigma) \pm \frac{1}{2} \sin 2\theta(s, \sigma) \right\}$$

Using a different analysis he gets that $\|\bar{T}_1\| < 1$

for $0 < \alpha \leq \pi$. In fact, for the above operators, the previous analysis gives identical results. However, Magnaradze then claims that since the norm of the above operators are individually less than one, this implies that the norm of T_1 is also less than one. He seems to be using the following incorrect reasoning: When S is located in the neighborhood of s_0 the only non-small contribution to the norm of T_1 is from that part of the domain of integration in which $\sigma: [s_0-\epsilon, s_0+\epsilon]$. Likewise, when S is located in the neighborhood of s_0+l , the only non $O(\epsilon)$ contribution to T_1 is from $\sigma: [s_0+l-\epsilon, s_0+l+\epsilon]$. But this is not the case. There is a finite non-zero contribution to the norm of T_1 for S in the neighborhood of s_0 from both $\sigma: [s_0-\epsilon, s_0+\epsilon]$ and $\sigma: [s_0+l-\epsilon, s_0+l+\epsilon]$.

Section 7. An Expression for the Kernel

An explicit form for the kernel of the integral operator of eqn (8.26) is derived. This is given by:

$$(1 - T_1)^{-1} V\Phi(s) = \left(\sum_{m=0}^{\infty} T_1^m \right) V\Phi(s). \quad (8.27)$$

The first term in the series is:

$$V\Phi(s) = \frac{1}{\pi} \int_{\xi - \xi_{s_0\epsilon}}^{\xi} \Phi(\sigma) d\sigma \zeta(s, \sigma)$$

The second term in the series is:

$$T_1 V\Phi(s) = \frac{1}{\pi} \int_{\xi_{s_0\epsilon}}^{\xi} \int_{\xi - \xi_{s_0\epsilon}}^{\xi} \Phi(\sigma) I_0(\bar{\sigma}, \sigma) d\sigma d\bar{\sigma} \zeta(s, \bar{\sigma}) \quad (8.28)$$

where

$$I_0(\bar{\sigma}, \sigma) \equiv \frac{1}{\pi} \frac{\partial \zeta(\bar{\sigma}, \sigma)}{\partial \sigma}.$$

The term $I_0(\bar{\sigma}, \sigma)$ is well defined and uniformly bounded by a number, say, M over the domain σ restricted to

$\xi - \xi_{s_0\epsilon}$. The orders of integration in eqn (8.28) are

interchanged. In order to do this the following theorem is used (Hobson, 1957; p.561):

"The necessary and sufficient condition that the bounded function $f(x^{(1)}, x^{(2)})$ should possess a Riemann-Stieltjes-integral with respect to the function, $\phi(x^{(1)}, x^{(2)})$, of bounded variation, in accordance with the definition in §254

, is that the variation of $\phi(\chi^{(1)}, \chi^{(2)})$ over the set of points of discontinuity of $f(\chi^{(1)}, \chi^{(2)})$ should be zero."

The function $\gamma(s, \bar{\sigma})$ is of bounded variation if we have

$$\int_{\mathcal{C}} |d_{\sigma} \theta(s, \sigma)| < G < \infty \quad \text{FOR } s \in \mathcal{C}$$

That is to say that the contour $\partial \mathcal{D}$ is of "bounded rotation". This is a consequence of assuming that $\chi(s)$, the curvature of $\partial \mathcal{D}$, is Hölder continuous, and that $\partial \mathcal{D}$ has finite length.

It should be kept in mind that the integral written symbolically as $\int_{\mathcal{C} - \mathcal{C}_{s_0, \epsilon}}$ is in fact

$$\int_{(\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l)} + \int_{(\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [l, 2l)} \quad \text{as a consequence of the}$$

way the domain has been extended from $[0, l)$ to $[0, 2l)$. This means that the "apparent" simultaneous discontinuity in both the functions at $\sigma = l$ is an artifact. Hence, the proper interpretation of the integral $\int_{\mathcal{C} - \mathcal{C}_{s_0, \epsilon}}$ as the integration variable travels through 0 or l is that of an improper integral.

Since the functions $\Gamma_0(\bar{\sigma}, \sigma)$, and $\Phi(\sigma)$ are continuous over the two sets of domains

$$\sigma \in (\xi - \xi_{s_0 \epsilon}) \cap [0, \ell); \quad \bar{\sigma} \in \xi_{s_0 \epsilon} \quad \text{and}$$

$$\sigma \in (\xi - \xi_{s_0 \epsilon}) \cap [0, 2\ell); \quad \bar{\sigma} \in \xi_{s_0 \epsilon}$$

the orders of integration can be interchanged and eqn (8.28)

becomes:

$$T_1 V \Phi(s) = \frac{1}{\pi} \int_{\xi - \xi_{s_0 \epsilon}}^{\xi} \Phi(\sigma) \int_{\xi_{s_0 \epsilon}}^{\xi} I_0(\bar{\sigma}, \sigma) d\bar{\sigma} \chi(s, \bar{\sigma}) d\sigma \quad (8.29)$$

The quantity $I_1(s, \sigma)$ is defined as:

$$I_1(s, \sigma) \equiv \frac{1}{\pi} \int_{\xi_{s_0 \epsilon}}^{\xi} I_0(\bar{\sigma}, \sigma) d\bar{\sigma} \chi(s, \bar{\sigma})$$

The third term in eqn (8.27) is:

$$T_1^2 V \Phi(s) = \frac{1}{\pi} \int_{\xi_{s_0 \epsilon}}^{\xi} \int_{\xi - \xi_{s_0 \epsilon}}^{\xi} \Phi(\sigma) I_1(\bar{\sigma}, \sigma) d\sigma d\bar{\sigma} \chi(s, \bar{\sigma}) \quad (8.30)$$

where eqn (8.29) has been used. It is verified in section 8 that, indeed, the function $I_1(s, \sigma)$ is uniformly Hölder continuous in both of its variables as long as they are restricted to the following domain:

$$s \in \xi \quad ; \quad \sigma \in \xi - \xi_{s_0 \epsilon}$$

(Again it should be kept in mind that the function is not con-

tinuous across the lines $s=l$ and $\sigma=l$. However, the same Hölder constants hold for the "four subdomains".) In addition, the function $I_1(s, \sigma)$ is uniformly bounded in the above domain since

$$I_1(s, \sigma) \leq \frac{1}{\pi} \int_{\mathcal{C}_{s_0 \in}} |I_0(\bar{\sigma}, \sigma)| |d_{\bar{\sigma}} \zeta(s, \bar{\sigma})|$$

and for $\bar{\sigma} \in \mathcal{C}_{s_0 \in}$ and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0 \in}$:

$$|I_1(s, \sigma)| \leq \frac{M}{\pi} \int_{\mathcal{C}_{s_0 \in}} |d_{\bar{\sigma}} \zeta(s, \bar{\sigma})|$$

or

$$|I_1(s, \sigma)| \leq M$$

Using these results the orders of integration can be changed in eqn (8.30). The result of this is:

$$T_1^2 V \Phi(s) = \int_{\mathcal{C} - \mathcal{C}_{s_0 \in}} \Phi(\sigma) I_1(s, \sigma) d\sigma$$

By inductive reasoning, it is easily seen that if (i) the quantities $I_n(s, \sigma)$ are defined by:

$$I_n(s, \sigma) \equiv \frac{1}{\pi} \int_{\mathcal{C}_{s_0 \in}} I_{n-1}(\bar{\sigma}, \sigma) d_{\bar{\sigma}} \zeta(s, \bar{\sigma}) \quad (8.31)$$

and (ii), the fact that (shown in section 8) $I_n(s, \sigma)$ is

uniformly Hölder continuous in both S and σ for the same domain as mentioned for $I_1(s, \sigma)$, and that (iii) the quantities $I_n(s, \sigma)$ are uniformly bounded by M for S and σ in the domain of interest, then it follows that:

$$T^n V \Phi(s) = \int_{\rho - \rho_{s_0 \epsilon}} \Phi(\sigma) I_{n-1}(s, \sigma) d\sigma.$$

As a result of this, eqn (8.27) can be rewritten in the following form:

$$(1 - T_1)^{-1} V \Phi(s) = \sum_{n=0}^{\infty} \int_{\rho - \rho_{s_0 \epsilon}} \Phi(\sigma) I_n(s, \sigma) d\sigma. \quad (8.31)$$

A theorem due to Beppo Levi (Riesz and Nagy, 1955; p.36) can be used to change the order of summation and integration. This theorem states that if $f_1(x), f_2(x), \dots, f_n(x), \dots$ are \mathcal{L}' and $\sum_{n=1}^{\infty} \int_a^b |f_n(x)| dx < \infty$ then $\sum_{n=1}^{\infty} f_n(x)$ converges

absolutely a.e. and its sum, $f(x)$ defined by

$$f(x) \equiv \sum_{n=1}^{\infty} f_n(x) \quad \text{is also } \mathcal{L}' \text{ and}$$

$$\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx = \int_a^b f(x) dx.$$

Both $\Phi(\sigma)$ and $I_n(s, \sigma)$ are uniformly bounded,

and so

$$\sum_{n=0}^{\infty} \int_{\mathcal{C}-\mathcal{C}_{s_0\epsilon}} |\Phi(\sigma)| |I_n(s, \sigma)| d\sigma \leq \sup |\Phi(\sigma)| \sum_{n=0}^{\infty} \int_{\mathcal{C}-\mathcal{C}_{s_0\epsilon}} |I_n(s, \sigma)| d\sigma$$

$$\leq \sup |\Phi(\sigma)| \frac{M}{1 - \|T_1\|}$$

$$< \infty \quad \text{for } \alpha_0 < \alpha \leq \pi$$

As a consequence of the above theorem $I(s, \sigma)$ can be defined:

$$I(s, \sigma) \equiv \sum_{n=0}^{\infty} I_n(s, \sigma) \quad (8.32)$$

where $I(s, \sigma)$ exists everywhere and the series converge absolutely. Eqn (8.27) can now be written in the following form:

$$(1 - T_1)^{-1} V \Phi(s) = \int_{\mathcal{C}-\mathcal{C}_{s_0\epsilon}} \Phi(\sigma) I(s, \sigma) d\sigma,$$

where $I(s, \sigma)$ is called the kernel of the integral operator.

The integral equation (8.26) can thus be written as follows:

$$\Phi(s) - \int_{\mathcal{C}-\mathcal{C}_{s_0\epsilon}} \Phi(\sigma) I(s, \sigma) d\sigma = F(s) \quad (8.33)$$

where

$$F(s) \equiv (1 - T_1)^{-1} D(s).$$

Section 8. Hölder Continuity of the Kernel.

It is now shown by cases that the $I_m(s, \sigma)$ are Hölder continuous in both s and σ within the following four domains:

$$D_1 : s \in \mathcal{C} \cap [0, \ell) ; \sigma \in (\mathcal{C} - \mathcal{C}_{s_0 \in}) \cap [0, \ell) ;$$

$$D_2 : s \in \mathcal{C} \cap [0, \ell) , \sigma \in (\mathcal{C} - \mathcal{C}_{s_0 \in}) \cap [\ell, 2\ell) ;$$

$$D_3 : s \in \mathcal{C} \cap [\ell, 2\ell) , \sigma \in (\mathcal{C} - \mathcal{C}_{s_0 \in}) \cap [0, \ell) ;$$

$$D_4 : s \in \mathcal{C} \cap [\ell, 2\ell) , \sigma \in (\mathcal{C} - \mathcal{C}_{s_0 \in}) \cap [\ell, 2\ell) .$$

In D_1 ,

$$\begin{aligned} I_m(s_1, \sigma) - I_m(s_2, \sigma) &= \left\{ 1 + \cos 2\theta(s_1, \sigma) \right\} \frac{\partial \theta}{\partial \sigma}(s_1, \sigma) + \\ &\quad - \left\{ 1 + \cos 2\theta(s_2, \sigma) \right\} \frac{\partial \theta}{\partial \sigma}(s_2, \sigma) \\ &= \left\{ \frac{\partial \theta}{\partial \sigma}(s_1, \sigma) - \frac{\partial \theta}{\partial \sigma}(s_2, \sigma) \right\} \left\{ 1 + \cos 2\theta(s_1, \sigma) \right\} + \\ &\quad + \frac{\partial \theta}{\partial \sigma}(s_2, \sigma) \left\{ \cos 2\theta(s_1, \sigma) - \cos 2\theta(s_2, \sigma) \right\} . \end{aligned}$$

It is shown in appendix 2.1 that $\frac{\partial \theta}{\partial \sigma}(s, \sigma)$ and $\theta(s, \sigma)$ are Hölder continuous in both s and σ for $s \in [0, \ell]$ and $\sigma \in (\mathcal{C} - \mathcal{C}_{s_0 \in}) \cap [0, \ell)$. Since the functions $\cos 2\theta(s, \sigma)$ and $\sin 2\theta(s, \sigma)$ are Lipschitz continuous

with Lipschitz constant equal to two, it follows that

$\cos 2\theta(s, \sigma)$ and $\sin 2\theta(s, \sigma)$ are Hölder continuous. The following constants are defined:

A_1, μ_1 ARE THE HÖLDER CONSTANTS FOR $\frac{\partial \theta}{\partial \sigma}$,

A_2, μ_2 ARE THE HÖLDER CONSTANTS FOR $\theta(s, \sigma)$,

and B_1 is the bound for $\frac{\partial \theta}{\partial \sigma}$, all constants are defined with respect to the above mentioned domains for S and σ . These constants depend on ϵ . Hence it follows that:

$$\begin{aligned} |I_0(s_1, \sigma) - I_0(s_2, \sigma)| &\leq 2A_1 |s_1 - s_2|^{\mu_1} + 2B_1 A_2 |s_1 - s_2|^{\mu_2} \\ &\leq \left\{ 2A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{s_1 - s_2}{l} \right|^{\mu_3} \end{aligned}$$

where $\mu_3 \equiv \min(\mu_1, \mu_2)$ (recall $0 < \mu_1 \leq 1$ and $0 < \mu_2 \leq 1$)

In domain D_2 ,

$$\begin{aligned} |I_0(s_1, \sigma) - I_0(s_2, \sigma)| &= \sin 2\theta(s_1, \sigma) \left\{ \frac{\partial \theta(s_1, \sigma)}{\partial \sigma} - \frac{\partial \theta(s_2, \sigma)}{\partial \sigma} \right\} + \\ &+ \frac{\partial \theta}{\partial \sigma}(s_2, \sigma) \left\{ \sin 2\theta(s_1, \sigma) - \sin 2\theta(s_2, \sigma) \right\}. \end{aligned}$$

Using the constants defined above and the fact that

$$\theta(s, \sigma) \equiv \theta(s, \sigma + l)$$

it follows that

$$\begin{aligned} |I_0(s_1, \sigma) - I_0(s_2, \sigma)| &\leq A_1 |s_1 - s_2|^{\mu_1} + 2B_1 A_2 |s_1 - s_2|^{\mu_2} \\ &\leq \left\{ A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{s_1 - s_2}{l} \right|^{\mu_3}. \end{aligned}$$

Following the same procedure as above, it is easily seen that

$$|I_0(s_1, \sigma) - I_0(s_2, \sigma)| \leq \left\{ A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{s_1 - s_2}{l} \right|^{\mu_3}$$

in domain D_3 and

$$|I_0(s_1, \sigma) - I_0(s_2, \sigma)| \leq \left\{ 2A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{s_1 - s_2}{l} \right|^{\mu_3}$$

in domain D_4 . Hence,

$$|I_0(s_1, \sigma) - I_0(s_2, \sigma)| \leq \left\{ 2A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{s_1 - s_2}{l} \right|^{\mu_3}$$

where s_1, s_2 are both located in $[0, l)$ or
 $[l, 2l)$ and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0 \in}$.

As a consequence of $\frac{\partial \oplus}{\partial \sigma}$ and \oplus being Hölder continuous in both s and σ , it follows that

$$|I(s, \sigma_1) - I(s, \sigma_2)| \leq \left\{ 2A_1 l^{\mu_1} + 2B_1 A_2 l^{\mu_2} \right\} \left| \frac{\sigma_1 - \sigma_2}{l} \right|^{\mu_3}$$

for $0 \leq s \leq 2l$ and σ_1, σ_2 both in $(\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l)$
or $\sigma_1, \sigma_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [l, 2l)$.

Hence, it has been shown that $I_0(s, \sigma)$ is Hölder continuous
in both s and σ for all s, σ for which it is defined.
 $I_1(s, \sigma)$ is now examined.

$$I_1(s, \sigma) \equiv \frac{1}{\pi} \int_{\mathcal{C}_{s_0, \epsilon}} I_0(\bar{\sigma}, \sigma) d\bar{\sigma} \chi(s, \bar{\sigma}).$$

Note that $I_1(s, \sigma)$ is defined in the domain $0 \leq s \leq 2l$
 and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0, \epsilon}$.

It turns out that the proofs of Hölder continuity for
 both $s \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l)$ and

$s \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [l, 2l)$ are straight forward and so
 these are considered first.

When $s \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l)$ and
 $\sigma \in [0, l]$, Appendix 2.1 shows that $\frac{\partial \Theta}{\partial \sigma}$ and Θ
 are Hölder continuous in s and σ .

In fact the Hölder constants are the same as those
 defined above. It then follows that:

$$I_1(s, \sigma) = \frac{1}{\pi} \int_{\mathcal{C}_{s_0, \epsilon} \cap [0, l)} I_0(\bar{\sigma}, \sigma) \left\{ 1 + \infty \Theta(s, \bar{\sigma}) \right\} \frac{\partial \Theta}{\partial \sigma}(s, \bar{\sigma}) d\bar{\sigma} +$$

(CONTINUED ON THE
 NEXT PAGE)

$$+ \frac{1}{\pi} \int_{\mathcal{C}_{s_0} \cap [l, 2l)} I_0(\bar{\sigma}, \sigma) \left\{ \sin 2\Theta(s, \bar{\sigma}) \right\} \frac{\partial \Theta}{\partial \sigma}(s, \bar{\sigma}) d\bar{\sigma}.$$

This enables us to write:

$$\begin{aligned} I_1(s_1, \sigma) - I_1(s_2, \sigma) &= \frac{1}{\pi} \int_{\mathcal{C}_{s_0} \cap [0, l)} I_0(\bar{\sigma}, \sigma) \left\{ \left[\frac{\partial \Theta}{\partial \sigma}(s_1, \bar{\sigma}) - \frac{\partial \Theta}{\partial \sigma}(s_2, \bar{\sigma}) \right] + \right. \\ &\quad \left. + \cos 2\Theta(s_1, \bar{\sigma}) \left[\frac{\partial \Theta}{\partial \sigma}(s_1, \bar{\sigma}) - \frac{\partial \Theta}{\partial \sigma}(s_1, \bar{\sigma}) \right] + \frac{\partial \Theta}{\partial \sigma}(s_2, \bar{\sigma}) \left[\cos 2\Theta(s_1, \bar{\sigma}) \right. \right. \\ &\quad \left. \left. - \cos 2\Theta(s_2, \bar{\sigma}) \right] \right\} d\bar{\sigma} + \\ &\quad + \frac{1}{\pi} \int_{\mathcal{C}_{s_0} \cap [l, 2l)} I_0(\bar{\sigma}, \sigma) \left\{ \sin 2\Theta(s_1, \bar{\sigma}) \left[\frac{\partial \Theta}{\partial \sigma}(s_1, \bar{\sigma}) - \frac{\partial \Theta}{\partial \sigma}(s_2, \bar{\sigma}) \right] + \right. \\ &\quad \left. + \frac{\partial \Theta}{\partial \sigma}(s_2, \bar{\sigma}) \left[\sin 2\Theta(s_1, \bar{\sigma}) - \sin 2\Theta(s_2, \bar{\sigma}) \right] \right\} d\bar{\sigma}. \end{aligned}$$

Hence,

$$|I_1(s_1, \sigma) - I_1(s_2, \sigma)| \leq \frac{1}{\pi} \int_{\mathcal{C}_{s_1, \sigma} \cap [0, \ell)} |I_0(\bar{\sigma}, \sigma)| \left\{ \left[2A_1 \ell^{\mu_1} + 2B_1 A_2 \ell^{\mu_2} \right] \cdot \left| \frac{s_1 - s_2}{\ell} \right|^{\mu_3} \right\} d\bar{\sigma} +$$

$$+ \frac{1}{\pi} \int_{\mathcal{C}_{s_1, \sigma} \cap [\ell, 2\ell)} |I_0(\bar{\sigma}, \sigma)| \left[A_1 \ell^{\mu_1} + 2B_1 A_2 \ell^{\mu_2} \right] \left| \frac{s_1 - s_2}{\ell} \right|^{\mu_3} d\bar{\sigma} .$$

M was defined by

$$M \equiv \sup_{\sigma, \bar{\sigma}} |I_0(\bar{\sigma}, \sigma)|$$

It can then be written that

$$|I_1(s_1, \sigma) - I_1(s_2, \sigma)| \leq \frac{2M\ell}{\pi} \left[2A_1 \ell^{\mu_1} + 2B_1 A_2 \ell^{\mu_2} \right] \left| \frac{s_1 - s_2}{\ell} \right|^{\mu_3}$$

It is easily seen that this inequality also holds for

$s_1, s_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, 2\ell)$ as well. From the above analysis it is seen that

$$|I_m(s_1, \sigma) - I_m(s_2, \sigma)| \leq \frac{2\ell}{\pi} \sup_{\sigma, \bar{\sigma}} |I_{m-1}(\bar{\sigma}, \sigma)| \left[2A_1 \ell^{\mu_1} + 2B_1 A_2 \ell^{\mu_2} \right] \left| \frac{s_1 - s_2}{\ell} \right|^{\mu_3}$$

But

$$|I_m(s, \sigma)| \leq \int_{\mathcal{C}_{s_0, \epsilon}} |I_{m-1}(\bar{\sigma}, \sigma)| |d\bar{\sigma} \chi(s, \bar{\sigma})| .$$

Using inductive reasoning and the fact that

$$\int_{\mathcal{C}_{s_0, \epsilon}} |d_{\bar{\sigma}} \chi(s, \bar{\sigma})| \leq \|T_1\| \quad \text{it follows that}$$

$$\sup_{s, \bar{\sigma}} |I_m(s, \sigma)| \leq M \|T_1\|^m.$$

Hence

$$|I_m(s_1, \sigma) - I_m(s_2, \sigma)| \leq M \|T_1\|^m \frac{2\ell}{\pi} [2A_1 \ell^{M_1} + 2B_1 A_2 \ell^{M_2}] \left| \frac{s_1 - s_2}{\ell} \right|^{M_3}$$

FOR $m = 0, 1, 2, \dots$ AND

$$s_1, s_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, \ell) \quad \text{OR} \quad s_1, s_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [\ell, 2\ell).$$

It is now shown that $I_1(s, \sigma)$ is Hölder continuous in σ for $\sigma \in \mathcal{C} - \mathcal{C}_{s_0, \epsilon}$ and $s \in \mathcal{C}$. It can be written:

$$|I_1(s, \sigma_1) - I_1(s, \sigma_2)| \leq \frac{1}{\pi} \int_{\mathcal{C}_{s_0, \epsilon}} |I_0(\bar{\sigma}, \sigma_1) - I_0(\bar{\sigma}, \sigma_2)| |d_{\bar{\sigma}} \chi(s, \bar{\sigma})|$$

OR

$$|I_1(s, \sigma_1) - I_1(s, \sigma_2)| \leq \frac{1}{\pi} [2A_1 \ell^{M_1} + 2B_1 A_2 \ell^{M_2}] \left| \frac{\sigma_1 - \sigma_2}{\ell} \right|^{M_3} \|T_1\|$$

$$\text{FOR } \sigma_1, \sigma_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, \ell) \quad \text{OR} \quad \sigma_1, \sigma_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [\ell, 2\ell).$$

This follows from the fact that $\frac{\partial \mathcal{Q}}{\partial \sigma}$ and \mathcal{Q} are Hölder continuous in S and σ for $S \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l]$ and $\sigma \in [0, l]$. By inductive reasoning it follows that

$$|I_m(s, \sigma_1) - I_m(s, \sigma_2)| \leq \|T_1\| \frac{M}{\Pi} [2A_1 l^{M_1} + 2B_1 A_2 l^{M_2}] \left| \frac{\sigma_1 - \sigma_2}{l} \right|^{M_3}$$

FOR $S \in \mathcal{C} - \mathcal{C}_{s_0, \epsilon}$ AND $\sigma_1, \sigma_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [0, l]$ OR

$$\sigma_1, \sigma_2 \in (\mathcal{C} - \mathcal{C}_{s_0, \epsilon}) \cap [l, 2l].$$

The more subtle proof of Hölder continuity for $S \in \mathcal{C}_{s_0, \epsilon} \cap [0, l]$ and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0, \epsilon}$ begins now. It has already been shown that $\chi(s, \sigma)$ is of uniform bounded variation over the domain $\mathcal{C}_{s_0, \epsilon}$, i.e.

$$\int_{\mathcal{C}_{s_0, \epsilon}} |d\sigma \chi(s, \sigma)| < 1 \quad \text{FOR ALL } S \in \mathcal{C}.$$

This implies that there exist two sets of non-decreasing functions, each set consisting of two functions, called the positive and negative variations of $\chi(s, \sigma)$. These sets are

denoted by $\left(\begin{array}{c} \bar{\sigma} = \sigma \\ P \chi(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 - \epsilon \end{array} , \begin{array}{c} \bar{\sigma} = \sigma \\ P \chi(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 + l - \epsilon \end{array} \right)$ and

$\left(\begin{array}{c} \bar{\sigma} = \sigma \\ N \chi(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 - \epsilon \end{array} , \begin{array}{c} \bar{\sigma} = \sigma \\ N \chi(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 + l - \epsilon \end{array} \right)$ respectively.

$\gamma(s, \sigma)$ can be written in the following form (Hobson, 1927; p.329):

$$\gamma(s, \sigma) - \gamma(s, s_0 - \epsilon) = \int_{\bar{\sigma}=s_0-\epsilon}^{\bar{\sigma}=\sigma} P \gamma(s, \bar{\sigma}) - \int_{\bar{\sigma}=s_0-\epsilon}^{\bar{\sigma}=\sigma} N \gamma(s, \bar{\sigma})$$

For $\sigma \in]s_0, \epsilon \cap [0, l)$,

$$\gamma(s, \sigma) - \gamma(s, s_0 + l - \epsilon) = \int_{\bar{\sigma}=s_0+l-\epsilon}^{\bar{\sigma}=\sigma} P \gamma(s, \bar{\sigma}) - \int_{\bar{\sigma}=s_0+l-\epsilon}^{\bar{\sigma}=\sigma} N \gamma(s, \bar{\sigma})$$

For $\sigma \in]s_0, \epsilon \cap [l, 2l)$.

Substituting the above representation for $\gamma(s, \sigma)$ in the expression for $I_1(s, \sigma)$ gives:

$$\begin{aligned} \pi I_1(s, \sigma) = & \int_{]s_0, \epsilon \cap [0, l)} I_0(\bar{\sigma}, \sigma) d\bar{\sigma} \left[\int_{\bar{\sigma}=s_0-\epsilon}^{\bar{\sigma}=\bar{\sigma}} P \gamma(s, \bar{\sigma}) - \int_{\bar{\sigma}=s_0-\epsilon}^{\bar{\sigma}=\bar{\sigma}} N \gamma(s, \bar{\sigma}) \right] + \\ & + \int_{]s_0, \epsilon \cap [l, 2l)} I_0(\bar{\sigma}, \sigma) d\bar{\sigma} \left[\int_{\bar{\sigma}=s_0+l-\epsilon}^{\bar{\sigma}=\bar{\sigma}} P \gamma(s, \bar{\sigma}) - \int_{\bar{\sigma}=s_0+l-\epsilon}^{\bar{\sigma}=\bar{\sigma}} N \gamma(s, \bar{\sigma}) \right]. \end{aligned}$$

Using the Mean Value Theorem and the fact that $I_0(\bar{\sigma}, \sigma)$ is a continuous function in the above domains, it follows that

$$\begin{aligned} \pi I_1(s, \sigma) = & I_0(\sigma', \sigma) \underset{\bar{\sigma} = s_0 - \epsilon}{\overset{\bar{\sigma} = s_0 + \epsilon}{P}} \chi(s, \bar{\sigma}) - I_0(\sigma'', \sigma) \underset{\bar{\sigma} = s_0 - \epsilon}{\overset{\bar{\sigma} = s_0 + \epsilon}{N}} \chi(s, \bar{\sigma}) + \\ & + I_0(\bar{\sigma}', \sigma) \underset{\bar{\sigma} = s_0 + l - \epsilon}{\overset{\bar{\sigma} = s_0 + l + \epsilon}{P}} \chi(s, \bar{\sigma}) - I_0(\bar{\sigma}'', \sigma) \underset{\bar{\sigma} = s_0 + l - \epsilon}{\overset{\bar{\sigma} = s_0 + l + \epsilon}{N}} \chi(s, \bar{\sigma}) . \end{aligned}$$

WHERE σ' AND $\sigma'' \in \mathcal{C}_{s_0 \in} \cap [0, l)$,

AND $\bar{\sigma}'$ AND $\bar{\sigma}'' \in \mathcal{C}_{s_0 \in} \cap [l, 2l)$.

This equation can be rewritten in the following form:

$$\begin{aligned} \pi I_1(s, \sigma) = & I_0(\sigma'', \sigma) \left\{ \chi(s, s_0 + \epsilon) - \chi(s, s_0 - \epsilon) \right\} + \\ & + \underset{\bar{\sigma} = s_0 - \epsilon}{\overset{\bar{\sigma} = s_0 + \epsilon}{P}} \chi(s, \bar{\sigma}) \left\{ I_0(\sigma', \sigma) - I_0(\sigma'', \sigma) \right\} + \\ & + I_0(\bar{\sigma}'', \sigma) \left\{ \chi(s, s_0 + l + \epsilon) - \chi(s, s_0 + l - \epsilon) \right\} + \\ & + \underset{\bar{\sigma} = s_0 + l - \epsilon}{\overset{\bar{\sigma} = s_0 + l + \epsilon}{P}} \chi(s, \bar{\sigma}) \left\{ I_0(\bar{\sigma}', \sigma) - I_0(\bar{\sigma}'', \sigma) \right\} . \end{aligned}$$

When $s \in \mathcal{C}_{s_0 \in} \cap [0, l)$, it follows from the definition of $\chi(s, \sigma)$ given in section 4 that

$$\gamma(s, s_0 + \epsilon) = \oplus(s, s_0 + \epsilon) + \frac{1}{2} \sin 2 \oplus(s, s_0 + \epsilon)$$

$$\gamma(s, s_0 - \epsilon) = \oplus(s, s_0 - \epsilon) + \frac{1}{2} \sin 2 \oplus(s, s_0 - \epsilon)$$

$$\gamma(s, s_0 + l + \epsilon) = -\frac{1}{2} \cos 2 \oplus(s, s_0 + l + \epsilon)$$

$$\gamma(s, s_0 + l - \epsilon) = -\frac{1}{2} \cos 2 \oplus(s, s_0 + l - \epsilon)$$

If $s_1, s_2 \in \mathcal{C}_{s_0, \epsilon} \cap [0, l)$ or $s_1, s_2 \in \mathcal{C}_{s_0, \epsilon} \cap [l, 2l)$
then

$$|\gamma(s_1, s_0 + \epsilon) - \gamma(s_2, s_0 + \epsilon)| \leq 2A_2 |s_1 - s_2|^{M_2}$$

$$|\gamma(s_1, s_0 - \epsilon) - \gamma(s_2, s_0 - \epsilon)| \leq 2A_2 |s_1 - s_2|^{M_2}$$

$$|\gamma(s_1, s_0 + l + \epsilon) - \gamma(s_2, s_0 + l + \epsilon)| \leq 2A_2 |s_1 - s_2|^{M_2}$$

$$|\gamma(s_1, s_0 + l - \epsilon) - \gamma(s_2, s_0 + l - \epsilon)| \leq 2A_2 |s_1 - s_2|^{M_2}$$

All that remains is to show that $\begin{matrix} \bar{\sigma} = s_0 + \epsilon \\ P \gamma(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 - \epsilon \end{matrix}$ and

$\begin{matrix} \bar{\sigma} = s_0 + l + \epsilon \\ P \gamma(s, \bar{\sigma}) \\ \bar{\sigma} = s_0 + l - \epsilon \end{matrix}$ are Hölder continuous. One consequence of assuming that $\mathcal{K}(s_0^+)$ and $\mathcal{K}(s_0^-)$, the one sided curvatures of $\partial\mathcal{D}$ at the corner, exist is that as $s \rightarrow s_0$ from either direction, the curve can be

approximated by an arc of a circle with radius $R_1 \equiv \frac{1}{\chi(s_0^-)}$
 or $R_1 \equiv \frac{1}{\chi(s_0^+)}$. There are four possibilities for the
 geometry of $\partial\Omega$ in the neighborhood of s_0 . They are shown
 in figure (8.5).

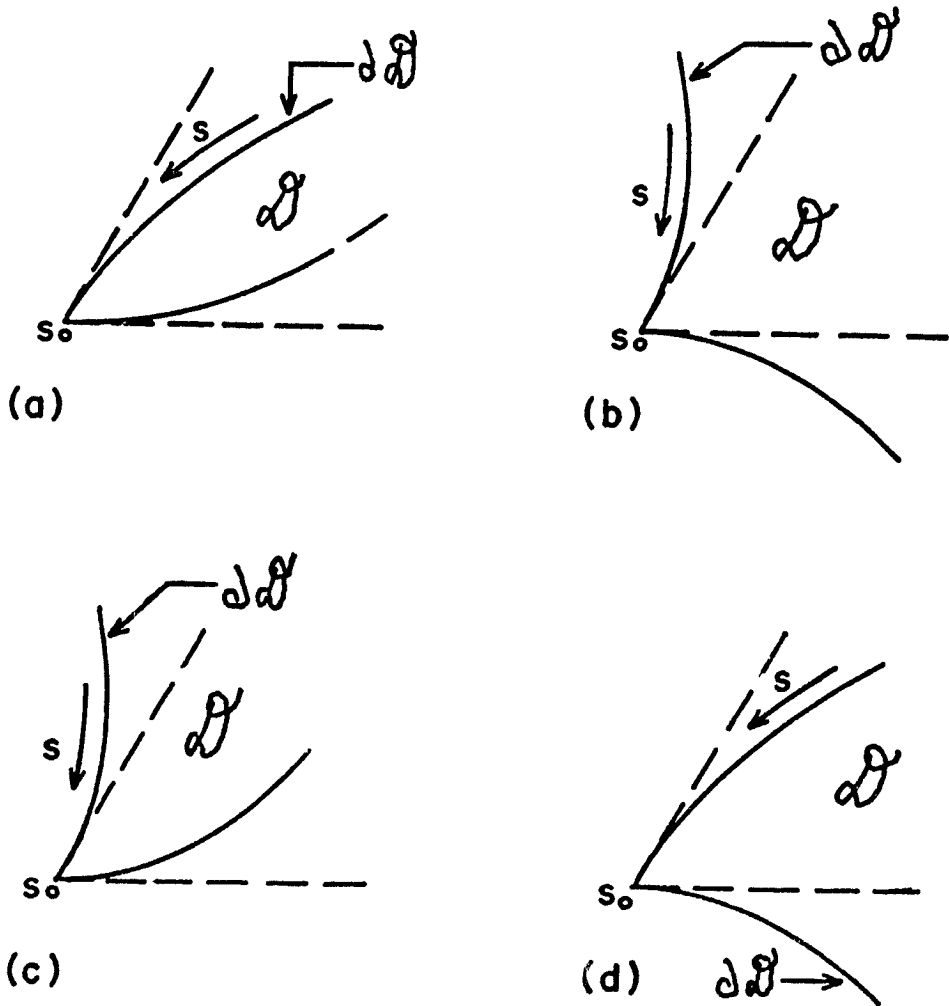


Fig. (8.5) The four possible geometries of $\partial\Omega$ in the neighborhood of s_0 .

Without loss in generality we can define $\theta(s, \sigma)$ as indicated in figure (8.6).

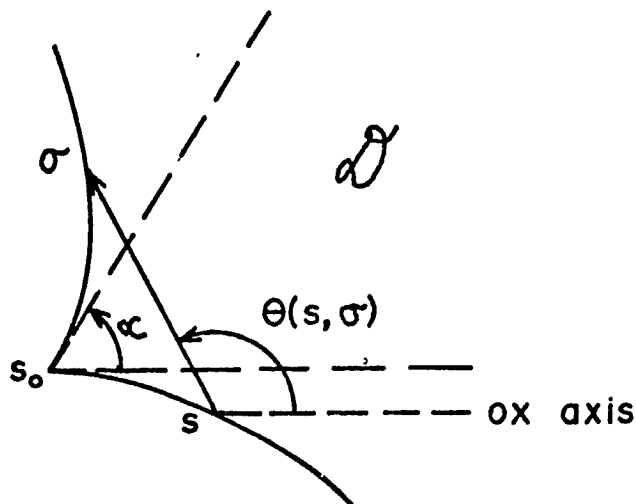


Fig. (8.6) The OX axis is chosen to be parallel to $\zeta(s_0^+)$.

For case (a), we plot $\oplus(s, \sigma)$ verse σ for $s < s_0$, $s = s_0$ and $s > s_0$, refer to figure (8.7). Regardless where s is located in $\mathcal{P}_{s_0 \in}$, the function $\oplus(s, \sigma)$ is a non-decreasing function of σ . We also have that when $s, \sigma \in \mathcal{P}_{s_0 \in} \cap [0, \ell)$ that

$$\zeta(s, \sigma) = \oplus(s, \sigma) + \frac{1}{2} \sin 2\oplus(s, \sigma) \quad . \quad \text{It is}$$

easily seen that $\zeta(s, \sigma)$ is a non-decreasing function of σ . This means that the positive variation of $\zeta(s, \sigma)$ has the following simple form:

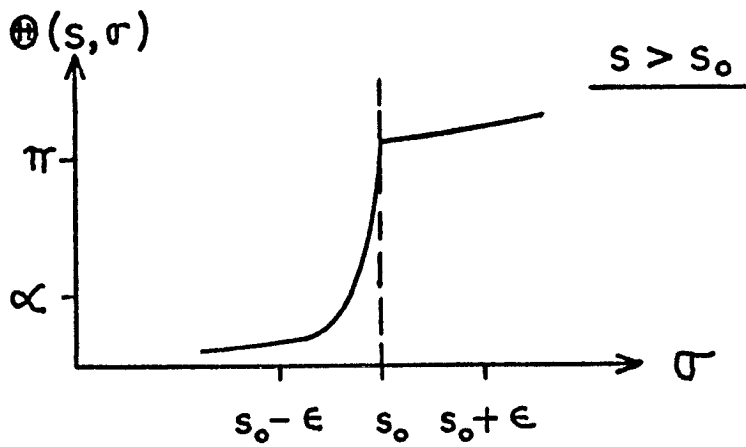
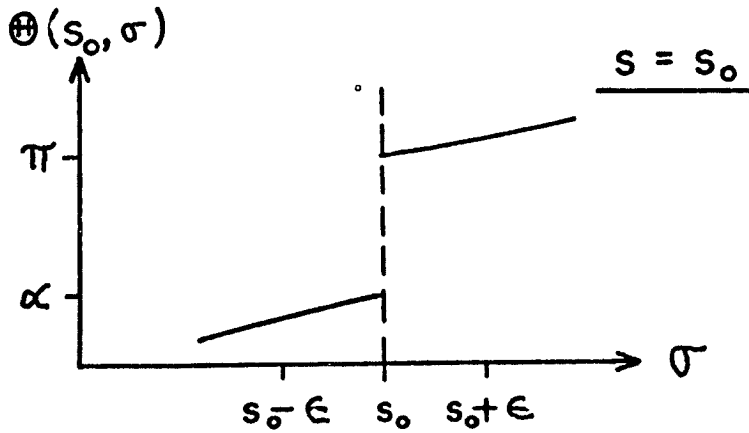
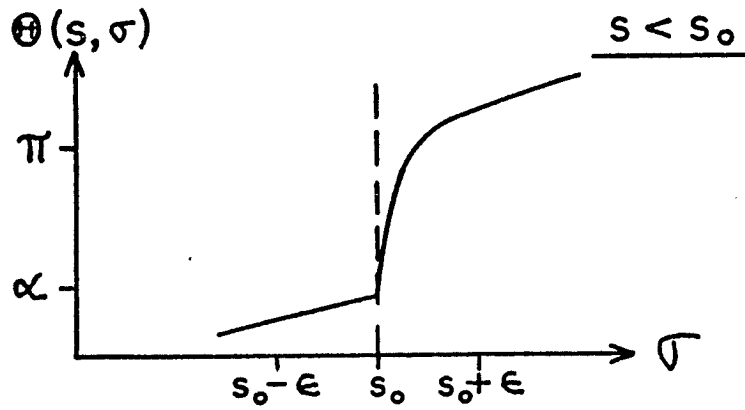


Fig. (8.7) Qualitative description of $\Theta(s, \sigma)$ for $s < s_0$, $s = s_0$ and $s > s_0$, in the domain $\sigma \in [s_0 - \epsilon, s_0 + \epsilon]$. This is for case (a).

$$\begin{aligned}
 \sigma = s_0 + \epsilon \\
 \sigma = s_0 - \epsilon \\
 P \chi(s, \sigma) &= \oplus(s, s_0 + \epsilon) - \oplus(s, s_0 - \epsilon) + \\
 &+ \frac{1}{2} \left\{ \sin 2 \oplus(s, s_0 + \epsilon) - \sin 2 \oplus(s, s_0 - \epsilon) \right\}
 \end{aligned}$$

This gives:

$$\begin{aligned}
 \left| \begin{array}{c} \sigma = s_0 + \epsilon \\ P \chi(s_1, \sigma) \\ \sigma = s_0 - \epsilon \end{array} - \begin{array}{c} \sigma = s_0 + \epsilon \\ P \chi(s_2, \sigma) \\ \sigma = s_0 - \epsilon \end{array} \right| &\leq \left| \oplus(s_1, s_0 + \epsilon) - \oplus(s_2, s_0 + \epsilon) \right| + \\
 &+ \left| \oplus(s_1, s_0 - \epsilon) - \oplus(s_2, s_0 - \epsilon) \right| + \frac{1}{2} \left| \sin 2 \oplus(s_1, s_0 + \epsilon) + \right. \\
 &\left. - \sin 2 \oplus(s_2, s_0 + \epsilon) \right| + \frac{1}{2} \left| \sin 2 \oplus(s_1, s_0 - \epsilon) - \sin 2 \oplus(s_2, s_0 - \epsilon) \right|
 \end{aligned}$$

$$\left| \begin{array}{c} \sigma = s_0 + \epsilon \\ P \chi(s_1, \sigma) \\ \sigma = s_0 - \epsilon \end{array} - \begin{array}{c} \sigma = s_0 + \epsilon \\ P \chi(s_2, \sigma) \\ \sigma = s_0 - \epsilon \end{array} \right| \leq 4A_2 |s_1 - s_2|^{M_2}.$$

In a similar fashion, since $\chi(s, \sigma) = \oplus - \frac{1}{2} \sin 2 \oplus$

is a non-decreasing function of \oplus for $\sigma, s \in \mathcal{P}_{s_0, \epsilon} \cap [l, 2l]$

it follows that

$$\left| P \chi(s_1, \sigma) - P \chi(s_2, \sigma) \right| \leq 4A_2 |s_1 - s_2|^{M_2} .$$

$\begin{matrix} \sigma = s_0 + l + \epsilon & \sigma = s_0 + l + \epsilon \\ \sigma = s_0 + l - \epsilon & \sigma = s_0 + l - \epsilon \end{matrix}$

When $s \in \mathcal{P}_{s_0, \epsilon} \cap [0, l)$ and $\sigma \in \mathcal{P}_{s_0, \epsilon} \cap [0, l)$,

$\chi(s, \sigma) = -\frac{1}{2} \cos 2 \oplus$. Since $\oplus(s, \sigma)$ is a non-decreasing function of σ ,

$$P \chi(s, \sigma) = P \left\{ \left[\frac{1}{2} \cos 2 \oplus(s, \sigma) \right] \mu \left(\frac{\pi}{2} - \oplus(s, \sigma) \right) + \frac{1}{2} \mu \left(\oplus(s, \sigma) - \frac{\pi}{2} \right) \right\}$$

$\begin{matrix} \sigma = s_0 + \epsilon & \sigma = s_0 + \epsilon \\ \sigma = s_0 - \epsilon & \sigma = s_0 - \epsilon \end{matrix}$

as shown in figure (8.8). It is easily seen that:

$$\left| P \chi(s_1, \sigma) - P \chi(s_2, \sigma) \right| < 2A_2 |s_1 - s_2|^{M_2} .$$

$\begin{matrix} \sigma = s_0 + \epsilon & \sigma = s_0 + \epsilon \\ \sigma = s_0 - \epsilon & \sigma = s_0 - \epsilon \end{matrix}$

The same bound is obtained if $s \in \mathcal{P}_{s_0, \epsilon} \cap [l, 2l)$ and $\sigma \in \mathcal{P}_{s_0, \epsilon} \cap [0, l)$:

$$\left| P \chi(s_1, \sigma) - P \chi(s_2, \sigma) \right| < 2A_2 |s_1 - s_2|^{M_2} .$$

$\begin{matrix} \sigma = s_0 + l + \epsilon & \sigma = s_0 + l + \epsilon \\ \sigma = s_0 + l - \epsilon & \sigma = s_0 + l - \epsilon \end{matrix}$

It has been shown for case (a) that $I_1(s, \sigma)$ is Hölder continuous in s for $s \in \mathcal{P}_{s_0, \epsilon}$ and

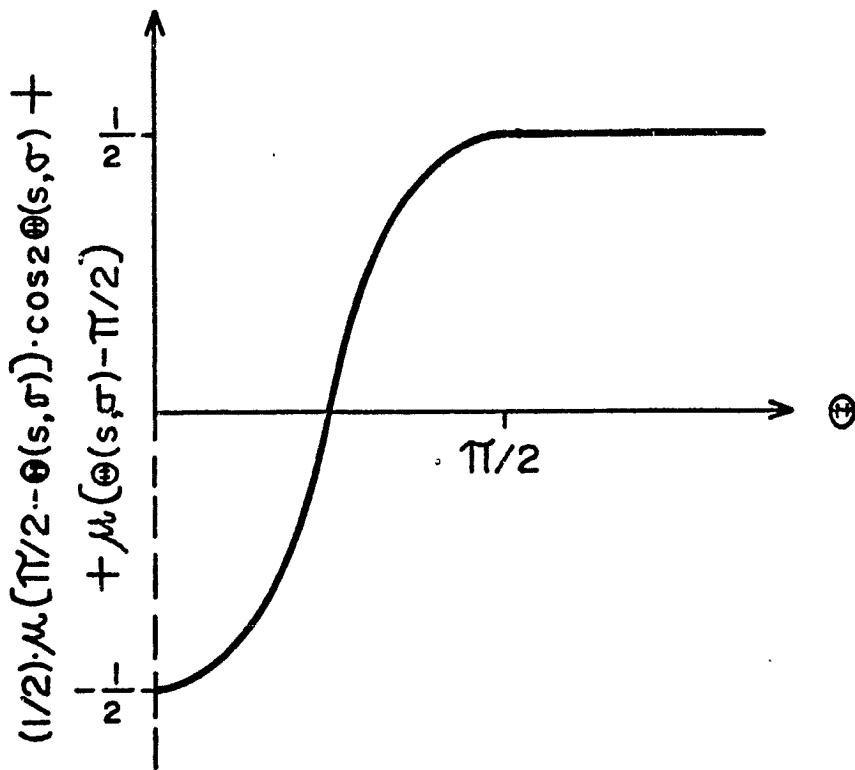


Fig. (8.8) The positive variation of the function $\frac{1}{2} \mu(2\theta)$ versus θ .

$$\sigma \in (\rho - \rho_{s_0 \epsilon}) \cap [0, \rho) \quad \text{OR} \quad \sigma \in (\rho - \rho_{s_0 \epsilon}) \cap [\rho, 2\rho).$$

Since the analysis for case (b) is a little different than for case (a), it is now examined. In this case $\theta(s, \sigma)$ is not a monotonic function of σ . Figure (8.9) illustrates

$$\theta(s, \sigma) \quad \text{for} \quad s < s_0, \quad s = s_0 \quad \text{and} \quad s > s_0.$$

If $\sigma, s \in \rho_{s_0 \epsilon} \cap [0, \rho)$ is considered, then

$$\chi(s, \sigma) = \theta(s, \sigma) + \frac{1}{2} \mu(2\theta(s, \sigma)) \quad \text{which is a non-}$$

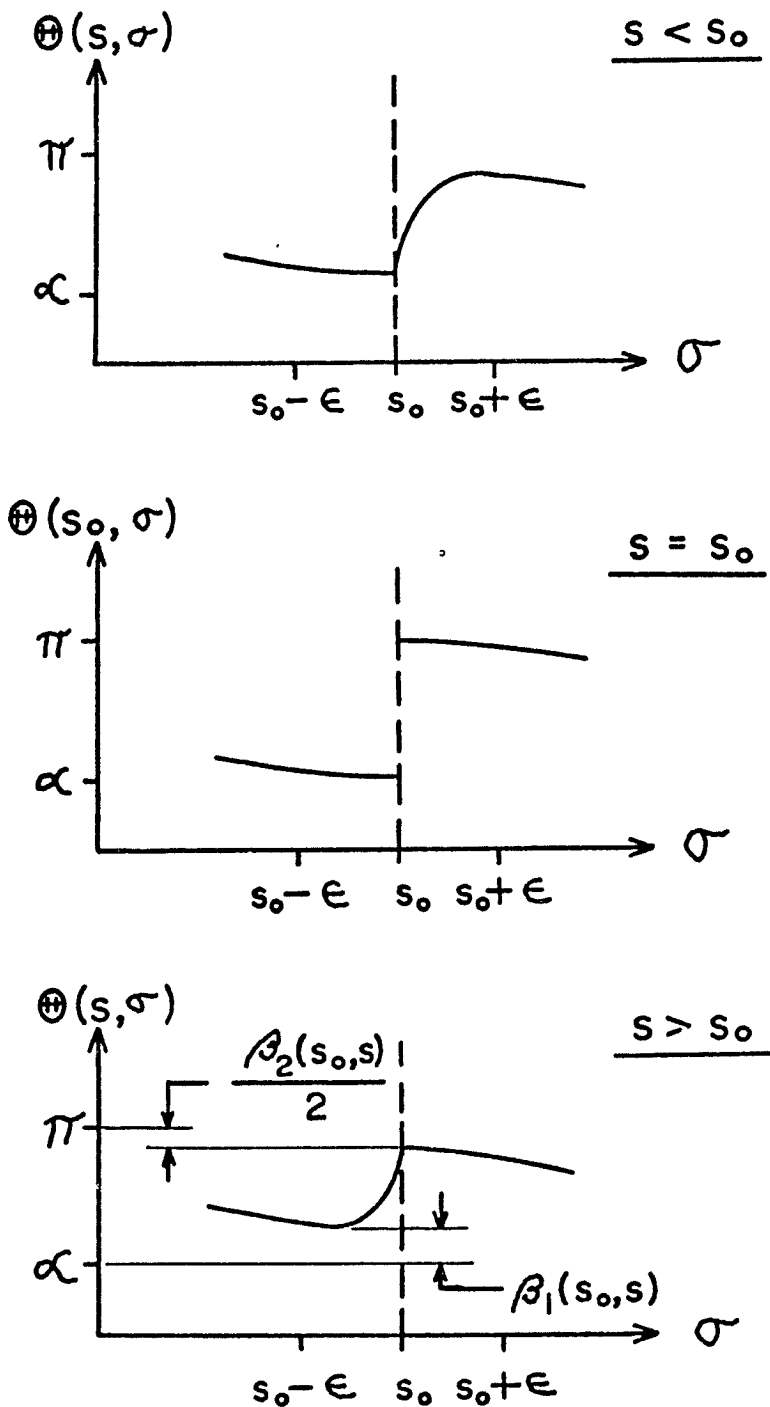


Fig. (8.9) This is a qualitative description of $\Theta(s, \sigma)$ for $s < s_0$, $s = s_0$ and $s > s_0$ for σ in the interval $[s_0 - \epsilon, s_0 + \epsilon]$. This is for case (c).

decreasing function of $\oplus(s, \sigma)$. This means there is no contribution to the positive variation of $\chi(s, \sigma)$ when \oplus decreases in value. The angles $\oplus_M(s)$ and $\oplus_m(s)$ are defined as the maximum and minimum values of $\oplus(s, \sigma)$ when

$\sigma \in \mathcal{C}_{s_0 \pm \epsilon}$. It can then be written that:

$$\left| \begin{matrix} \sigma = s_0 + \epsilon \\ \sigma = s_0 - \epsilon \end{matrix} P \chi(s_1, \sigma) - \begin{matrix} \sigma = s_0 + \epsilon \\ \sigma = s_0 - \epsilon \end{matrix} P \chi(s_2, \sigma) \right| \leq \lambda \left| \oplus_m(s_1) - \oplus_m(s_2) \right| + \lambda \left| \oplus_M(s_1) - \oplus_M(s_2) \right|,$$

where $\oplus_m(s_1)$ is the minimum value of $\oplus(s, \sigma)$ for $\sigma \in \mathcal{C}_{s_0 \pm \epsilon}$ and $\oplus_M(s_1)$ is the maximum value of $\oplus(s, \sigma)$ for $\sigma \in \mathcal{C}_{s_0 \pm \epsilon}$. For the case when both s_1 and s_2 are greater than s_0 there exist three possibilities for $\oplus_m(s_1) - \oplus_m(s_2)$:

$$(i) \quad \left| \oplus_m(s_1) - \oplus_m(s_2) \right| = \left| \oplus(s_1, s_0 - \epsilon) - \oplus(s_2, s_0 - \epsilon) \right|$$

$$\leq A_2 |s_1 - s_2|^{M_2}$$

$$(ii) \quad \left| \oplus_m(s_1) - \oplus_m(s_2) \right| = \left| \beta_1(s_1, s_2) \right|$$

where $\beta_1(s_1, s_2)$ is the difference in minimum values reached by $\oplus(s_1, \sigma)$ and $\oplus(s_2, \sigma)$ for $\sigma \in \mathcal{C}_{s_0 \pm \epsilon}$

It can be shown for a sufficiently small choice of ϵ

that $\beta_1(s_1, s_2) < \left[\frac{2 \sin \alpha}{R_1} \right]^{1/2} |s_1 - s_2|^{1/2}$ [Refer

to the equation derived in appendix 2.2 (B.I.Dussan V., 1972).]

(iii) $|\Theta_m(s_1) - \Theta_m(s_2)| = |\Theta(s_1, s_0 - \epsilon) - \alpha - \beta_1(s_0, s_2)|$
 where it is assumed here that $s_1 > s_2$. There exists \bar{s} where $s_1 > \bar{s} > s_2$ such that the minimum in the $\Theta(\bar{s}, \sigma)$ curve occurs at $\sigma = s_0 - \epsilon$. This enables one to write:

$$\begin{aligned} |\Theta_m(s_1) - \Theta_m(s_2)| &\leq |\Theta(s_2, s_0 - \epsilon) - \Theta(\bar{s}, s_0 - \epsilon)| + |\beta_1(\bar{s}, s_1)| \\ &< A_2 |s_2 - \bar{s}|^{\mu_2} + \left[\frac{2 \sin \alpha}{R_1} \right]^{\frac{1}{2}} |s_1 - \bar{s}|^{\frac{1}{2}} \\ &< A_2 |s_2 - s_1|^{\mu_2} + \left[\frac{2 \sin \alpha}{R_1} \right]^{\frac{1}{2}} |s_1 - s_2|^{\frac{1}{2}} \\ &< \left[A_2 l^{\mu_2} + \left\{ \frac{2 \sin \alpha}{R_1} \right\}^{\frac{1}{2}} l^{\frac{1}{2}} \right] \left| \frac{s_1 - s_2}{l} \right|^{\mu_5} \end{aligned}$$

WHERE $\mu_5 \equiv \min(\frac{1}{2}, \mu_2)$

For the case when s_1 and s_2 are both less than s_0 we have:

$$\begin{aligned} |\Theta_m(s_1) - \Theta_m(s_2)| &= |\Theta(s_1, s_0) - \Theta(s_2, s_0)| \\ &\leq A_2 |s_1 - s_2|^{\mu_2} \end{aligned}$$

For the case when $s_1 > s_0$ and $s_2 < s_0$ there exist two possibilities:

(i)

$$\begin{aligned}
 |\oplus_m(s_1) - \oplus_m(s_2)| &= |\beta_1(s_0, s_1) + \alpha - \oplus(s_2, s_0)| \\
 &< \left[\frac{2 \sin \alpha}{R_1} \right]^{1/2} |s_1 - s_0|^{1/2} + A_2 |s_0 - s_2|^{M_2} \\
 &< \left[\left\{ \frac{2 \sin \alpha}{R_1} \right\}^{1/2} l^{1/2} + A_2 l^{M_2} \right] \left| \frac{s_1 - s_2}{l} \right|^{M_5}
 \end{aligned}$$

(ii)

$$\begin{aligned}
 |\oplus_m(s_1) - \oplus_m(s_2)| &< |\oplus(s_1, s_0 - \epsilon) - \oplus(s_2, s_0)| \\
 &< |\oplus(s_1, s_0 - \epsilon) - \oplus(s_0, s_0^-)| + |\oplus(s_0, s_0^-) - \oplus(s_0, s_2)| \\
 &< |\oplus(s_1, s_0 - \epsilon) - \oplus(\bar{s}, s_0 - \epsilon)| + |\beta(s_0, \bar{s})| + \\
 &\quad + |\oplus(s_0, s_0^-) - \oplus(s_0, s_2)|
 \end{aligned}$$

where

$$s_0 < \bar{s} < s_1 \quad \text{AND} \quad \oplus_m(\bar{s}) = \oplus(\bar{s}, s_0 - \epsilon)$$

$$\begin{aligned}
 |\oplus_m(s_1) - \oplus_m(s_2)| &< A_2 |s_1 - \bar{s}|^{M_2} + \left(\frac{2 \sin \alpha}{R_1} \right)^{1/2} |s_0 - \bar{s}|^{1/2} + \\
 &\quad + A_2 |s_0 - s_2|^{M_2}
 \end{aligned}$$

$$< \left[2A_2 l^{M_2} + \left(\frac{2 \sin \alpha}{R_1} \right)^{1/2} l^{1/2} \right] \left| \frac{s_1 - s_2}{l} \right|^{M_5}$$

By identical arguments (R_1 and R_2 change places) it can be shown that $|\oplus_M(s_1) - \oplus_M(s_2)|$ is Hölder continuous.

It follows that $\int_{\sigma=s_0-\epsilon}^{\sigma=s_0+\epsilon} \gamma(s, \sigma)$ is Hölder continuous for

$s \in \mathcal{C}_{s_0, \epsilon} \cap [0, l)$. It follows by identical reasoning that $\int_{\sigma=s_0+l-\epsilon}^{\sigma=s_0+l+\epsilon} \gamma(s, \sigma)$ is also Hölder continuous for

$$s \in \mathcal{C}_{s_0, \epsilon} \cap [l, 2l).$$

When σ and s are in the remaining domains then

$$\gamma(s, \sigma) = -\frac{1}{2} C_{002} \oplus(s, \sigma).$$

There are two contributions to the positive variation of

$\gamma(s, \sigma)$. The first occurs when $\gamma(s, \sigma)$ is an increasing function of \oplus and \oplus is an increasing function of σ . This part is analogous to the previous discussions.

The second occurs when $\gamma(s, \sigma)$ is a decreasing function of \oplus and \oplus is a decreasing function of σ . It can be shown, using the equation for $\beta_1(s_1, s_2)$ when both s_1 and s_2 are larger than s_0 , that $\int_{\sigma=s_0-\epsilon}^{\sigma=s_0+\epsilon} \gamma(s, \sigma)$ for

$s \in \mathcal{C}_{s_0, \epsilon} \cap [l, 2l)$ and $\int_{\sigma=s_0+l-\epsilon}^{\sigma=s_0+l+\epsilon} \gamma(s, \sigma)$ for $s \in \mathcal{C}_{s_0, \epsilon} \cap [0, l)$ are Hölder continuous.

Cases (c) and (d) are similar to (a) and (b), requiring the same calculations; suffice it to say that the positive variation of γ can be shown to be Hölder continuous for

$s \in \mathcal{C}_{s_0 \in}$. Let us denote A and μ as its Hölder constants.

This gives:

$$|I_1(s_1, \sigma) - I_1(s_2, \sigma)| \leq \sup |I_0| \frac{8A_2}{\pi} |s_1 - s_2|^{\mu_2} + \sup |I_0| \frac{4A}{\pi} |s_1 - s_2|^{\mu}$$

By inductive reasoning we also get:

$$|I_m(s_1, \sigma) - I_m(s_2, \sigma)| \leq \frac{M}{\pi} \|T_1\|^{m-1} [8A_2 \ell^{\mu_2} + 4A \ell^{\mu}] \left| \frac{s_1 - s_2}{\ell} \right|^{\mu_4}$$

where $\mu_4 \equiv \min(\mu_2, \mu)$

s_1 and s_2 are both $\in \mathcal{C}_{s_0 \in} \cap [0, \ell)$ or

$\mathcal{C}_{s_0 \in} \cap [\ell, 2\ell)$, and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0 \in}$. Recall

that it has been already shown that I_n is Hölder continuous in σ for $s \in \mathcal{C}$ and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0 \in}$.

As a direct consequence of the Hölder constant for I_n being proportional to $\|T_1\|^n$ where $\|T_1\| < 1$ we have that $I(s, \sigma)$ is also Hölder continuous in σ and s for $s \in \mathcal{C}$ and $\sigma \in \mathcal{C} - \mathcal{C}_{s_0 \in}$.

This implies that we have an integral equation:

$$\Phi(s) - \int_{\mathcal{C} - \mathcal{C}_{s_0 \in}} \Phi(\sigma) I(s, \sigma) d\sigma = F(s)$$

The definition of $I(s, \sigma)$ is extended so that the above equation looks like a Fredholm equation:

$$\bar{I}(s, \sigma) \equiv \begin{cases} I(s, \sigma) & \text{for } \sigma \in \mathcal{L} - \mathcal{L}_{s_0} & \text{for all } s \\ 0 & \text{for } \sigma \in \mathcal{L}_{s_0} & \text{for all } s \end{cases}$$

The integral equation can be written in the form:

$$\Phi(s) - \int_0^{2l} \Phi(\sigma) \bar{I}(s, \sigma) d\sigma = F(s)$$

For any fixed s , the kernel $\bar{I}(s, \sigma)$ is uniformly bounded and piecewise continuous. For any fixed σ the kernel

$\bar{I}(s, \sigma)$ is Hölder continuous in s for s in $[0, l)$ or $(l, 2l)$. The function $F(s)$ is Hölder continuous since $D(s)$ is Hölder continuous; recall

$F(s) \equiv (1 - T_1)^{-1} D(s)$. (This follows using the same arguments when showing $I(s, \sigma)$ is Hölder continuous.)

Section 9. Hölder Continuity of the Solution.

Since the kernel $\bar{I}(s, \sigma)$ and $F(s)$ are \mathcal{L}^2 we can use the Fredholm Alternative (Reisz and Nagy, 1955, p.172). This states that if a solution exists, then it must be \mathcal{L}^2 . It is now shown that if a solution exists, then it must also be Hölder continuous for $s \in [0, \ell)$ and $s \in [\ell, 2\ell)$.

$$\Phi(s_1) - \Phi(s_2) = \int_0^{2\ell} \Phi(\sigma) [\bar{I}(s_1, \sigma) - \bar{I}(s_2, \sigma)] d\sigma + F(s_1) - F(s_2).$$

Using the Schwarz and triangle inequalities, it is found that

$$|\Phi(s_1) - \Phi(s_2)| \leq \left\{ \int_0^{2\ell} \Phi^2(\sigma) d\sigma \right\}^{1/2} \left\{ \int_0^{2\ell} [\bar{I}(s_1, \sigma) - \bar{I}(s_2, \sigma)]^2 d\sigma \right\}^{1/2} + |F(s_1) - F(s_2)|$$

It is known that (i) $\Phi(\sigma) \in \mathcal{L}^2$ and hence $\int_0^{2\ell} \Phi^2(\sigma) d\sigma$ is bounded, (ii) $\bar{I}(s, \sigma)$ is Hölder continuous, with Hölder constant independent of σ and (iii) $F(s)$ is Hölder continuous. Hence we know that $\Phi(s)$ for $s \in [0, \ell)$ and $\Phi(s)$ for $s \in [\ell, 2\ell)$ is Hölder continuous.

Note that the results are independent of the location of $S=0$ and so we can conclude that the functions $P(s)$ and $Q(s)$ are Hölder continuous on the entire contour ∂D .

It can be shown by the Fredholm Alternative that necessary and sufficient conditions for the existence of a solution $\varphi(z)$ is that:

$$\int_{\partial D} f(\zeta) d\zeta = 0$$

where $f(\zeta)$ is the complex velocity; refer to eqn (8.13). The proof follows almost word for word that of Mikhlin (1957; pp. 245-249) (also see Magnaradze, 1938b). This includes establishing the fact that $\varphi(z)$ can be analytically continued into ∞ , and hence gives rise to a solution to the biharmonic equation.

Section 10. Uniqueness of Solutions.

Finally the velocity field obtained from the above solution of $\varphi(z)$, is the only solution to the biharmonic equation when the boundary values of the velocity field are specified.

Let Ψ_1 and Ψ_2 represent solutions to the biharmonic equation in \mathcal{D} with $\nabla \Psi_1 = \nabla \Psi_2 = \nabla \Psi$ on $\partial \mathcal{D}$

It is shown that $\Psi_1 - \Psi_2 = \text{constant}$ in the domain \mathcal{D} .

Consider $\bar{\Psi}$,

$$\bar{\Psi} \equiv \Psi_1 - \Psi_2 .$$

It follows that $\nabla^4 \bar{\Psi} = 0$ in \mathcal{D} and $\nabla \bar{\Psi} = 0$ on $\partial \mathcal{D}$. If J is defined by the following:

$$J \equiv \int_{\mathcal{D}} \bar{\Psi} \nabla^4 \bar{\Psi} da ,$$

then since $\nabla^4 \bar{\Psi} = 0$, $J \equiv 0$. Using Green's theorem,

$$J = \int_{\partial \mathcal{D}} \bar{\Psi} \nabla(\nabla^2 \bar{\Psi}) \cdot \underline{n} dl - \int_{\mathcal{D}} \nabla(\nabla^2 \bar{\Psi}) \cdot \nabla \bar{\Psi} da$$

the boundary conditions $\nabla \bar{\Psi} = 0$ on $\partial \mathcal{D}$ imply that:

$$\bar{\Psi} = \text{constant on } \partial \mathcal{D}$$

Hence

$$\begin{aligned} \int_{\partial\Omega} \bar{\Psi} \nabla(\nabla^2 \bar{\Psi}) \cdot \underline{n} \, d\ell &= C \int_{\partial\Omega} \nabla(\nabla^2 \bar{\Psi}) \cdot \underline{n} \, d\ell \\ &= C \int_{\Omega} \nabla^4 \bar{\Psi} \, da = 0 \end{aligned}$$

This leaves:

$$J = - \int_{\Omega} \nabla(\nabla^2 \bar{\Psi}) \cdot \nabla \bar{\Psi} \, da .$$

Using Green's theorem again gives:

$$J = - \int_{\partial\Omega} \nabla^2 \bar{\Psi} \nabla \bar{\Psi} \cdot \underline{n} \, d\ell + \int_{\Omega} (\nabla^2 \bar{\Psi})^2 \, da .$$

However, $\nabla \bar{\Psi} \cdot \underline{n} = 0$ on $\partial\Omega$ this implies that

$$J = \int_{\Omega} (\nabla^2 \bar{\Psi})^2 \, da = 0$$

Since $(\nabla^2 \bar{\Psi})^2$ can never be negative, and since it has been shown that any solution to $\nabla^4 \bar{\Psi} = 0$ must be

C^∞ implies

$$\nabla^2 \bar{\Psi} = 0 \text{ in } \Omega$$

The solution to $\nabla^2 \bar{\Psi} = 0$ in Ω and $\bar{\Psi} = \text{constant}$ on $\partial\Omega$ is:

$$\bar{\Psi} = \text{constant in } \Omega$$

We can conclude that for a given velocity on the boundary, the solution of the biharmonic equation in terms of the velocity field is unique.

Appendix 1.

1.1. A function $h(x)$ is said to be absolutely continuous in (a, b) if, corresponding to an arbitrarily chosen positive number ϵ , another positive number δ can be so determined that, in every enumerable, or finite, set of non-overlapping intervals

$$(x_1, x_1'), (x_2, x_2'), \dots, (x_m, x_m'), \dots$$

in the interval (a, b) , and such that the total measure of the intervals is $< \delta$, the sum, or limiting sum

$\sum_r |h(x_r) - h(x_r')|$ is $< \epsilon$. (This definition is taken almost word for word from Hobson, 1927; pp.291 and 292).

1.2. It is clear that an absolutely continuous function is uniformly continuous (not the converse). All we have to do is consider one interval of length $< \delta$ instead of a set of intervals (Hobson, 1927; p.292). If $\omega(r, \theta)$ is a uniformly continuous function in r over the interval $(0, R]$ for $\theta = \theta_0$ then $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta_0}} \omega(r, \theta)$ exists (finite).

Demonstration:

The function $\omega(r, \theta_0)$ being uniformly continuous in r ... means: for any $\epsilon > 0$ there exists $\delta > 0$ independent of r , such that

$$|\omega(r_1, \theta_0) - \omega(r_2, \theta_0)| < \epsilon \quad \text{for } |r_1 - r_2| < \delta, \\ r_1 \text{ AND } r_2 \in (0, R].$$

Let us consider the special case in which $|r_1| < \eta$ and $|r_2| < \eta$, then restating the above we have:

for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$|w(r_1, \theta_0) - w(r_2, \theta_0)| < \epsilon \quad \text{for } |r_1| < \eta, |r_2| < \eta.$$

This is just the definition of $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta_0}} w(r, \theta)$.

Define a function $h(\theta)$:

$$h(\theta_0) \equiv \lim_{\substack{r \rightarrow 0 \\ \theta = \theta_0}} w(r, \theta).$$

1.3. If (a) $\frac{\partial \bar{u}}{\partial r}$ is monotonic without bound, or if (b) $|\frac{\partial \bar{u}}{\partial r}|$ is bounded, in both cases, in the neighborhood of $r=0$ on $\theta = \theta'$ where $0 < \theta' < \theta_1$, then

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} r \frac{\partial \bar{u}}{\partial r} = 0.$$

Demonstration:

By definition of \bar{u} , i.e. $\bar{u}(r, \theta) = u(r, \theta) - f(\theta)$ we have that $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \bar{u}(r, \theta) = 0$. Let us examine $\bar{u}(r, \theta)$

as a function of r holding θ fixed at θ' . In only this particular analysis define $\bar{u}(0, \theta') \equiv 0$. This makes $\bar{u}(r, \theta')$ an absolutely continuous function of r over the closed interval $[0, R]$. By the Radon-Nikodym theorem we can write:

$$\bar{u}(r, \theta') = \int_0^r \frac{\partial \bar{u}}{\partial r}(r, \theta') dr. \quad (1)$$

(a) If $\frac{\partial \bar{u}}{\partial r}$ is monotonic for sufficiently small r , without loss in generality it can be assumed $\frac{\partial \bar{u}}{\partial r} > 0$ in the region for $0 < r < D < R$. We also have

$$0 < r \frac{\partial \bar{u}}{\partial r} \Big|_r \leq \int_0^r \frac{\partial \bar{u}}{\partial r} (r, \theta') dr \quad \text{for any } 0 < r < D$$

Using eqn (1) gives:

$$0 < r \frac{\partial \bar{u}}{\partial r} \Big|_r \leq \bar{u} (r, \theta')$$

It is concluded since $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} \bar{u} (r, \theta) = 0$, hence

$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} r \frac{\partial \bar{u}}{\partial r} = 0.$$

(b) If $\left| \frac{\partial \bar{u}}{\partial r} \right|_r$ is bounded, say $\left| \frac{\partial \bar{u}}{\partial r} \right| < M$; then

$$\left| r \frac{\partial \bar{u}}{\partial r} \Big|_r \right| \leq rM$$

It is concluded since $\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} rM = 0$, hence

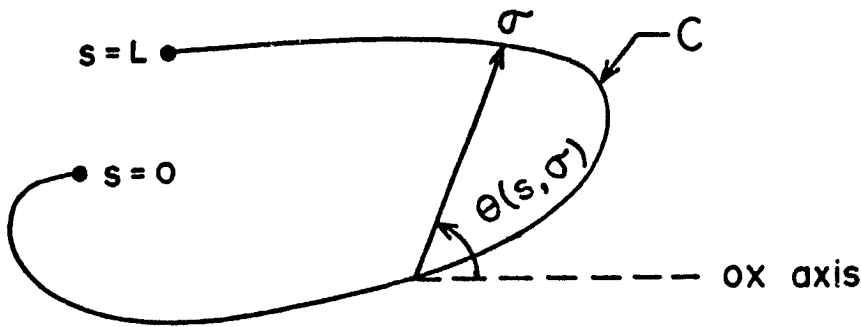
$$\lim_{\substack{r \rightarrow 0 \\ \theta = \theta'}} r \frac{\partial \bar{u}}{\partial r} = 0.$$

Appendix 2

2.1. Define the function $\Theta(s, \sigma)$ as follows:

$$\Theta(s, \sigma) \pmod{2\pi} = \theta(s, \sigma) + \pi \mu(s - \sigma)$$

where $\theta(s, \sigma)$ is the angle formed between the position vector pointing from $\gamma(s)$ to $\gamma(\sigma)$ and the Ox axis; refer to figure (2.1.1)



Fig(2.1.1)

The function $\theta(s, \sigma)$ is restricted to the interval $[0, 2\pi)$. It is easily seen that if the contour C is smooth (possesses a continuous tangent), then $\Theta(s, \sigma)$ is continuous for $s, \sigma \in [0, L]$. Muskhelishvili (1946; p.20) demonstrates that if the angle formed between the tangent vector to the contour and the Ox axis satisfies the Hölder condition, then $\theta(s, \sigma)$ also satisfies the Hölder condition in S ($0 < \sigma < L$) where S lies either on $[0, \sigma]$ or on $[\sigma, L]$. This implies that $\Theta(s, \sigma)$ must satisfy the Hölder condition. In addition since $\Theta(s, \sigma)$ is a continuous function of S

even as it passes through σ we have:

$$\Theta(s_1, \sigma) - \Theta(s_2, \sigma) = \Theta(s_1, \sigma) - \Theta(\sigma, \sigma) + \Theta(\sigma, \sigma) - \Theta(s_2, \sigma)$$

or

$$|\Theta(s_1, \sigma) - \Theta(s_2, \sigma)| \leq |\Theta(s_1, \sigma) - \Theta(\sigma, \sigma)| + |\Theta(\sigma, \sigma) - \Theta(s_2, \sigma)|$$

and hence $\Theta(s, \sigma)$ satisfies the Hölder condition in S or σ for any $s, \sigma \in [0, L]$. In a similar manner it can be shown (Muskhelishvili, 1949; p.21) that if the curvature of C satisfies the Hölder condition, then the function

$\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ satisfies the Hölder condition in S and σ for any $s, \sigma \in [0, L]$ if we define

$$\frac{\partial \Theta}{\partial \sigma}(\sigma, \sigma) = \frac{\partial \Theta}{\partial \sigma}(\sigma^+, \sigma) = \frac{\partial \Theta}{\partial \sigma}(\sigma^-, \sigma).$$

Keeping this in mind it is easily seen that $\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ also satisfies the Hölder condition in both S and σ for any $s, \sigma \in [0, L]$

The above results can be used to establish that if the curvature of $\partial \mathcal{D}$ satisfies the Hölder condition (at the corner where it satisfies the "one" sided Hölder condition), then

$\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ and $\Theta(s, \sigma)$ satisfy the Hölder condition

in S and σ for $s \in C$ and $\sigma \in C - C_{s_0 \in \epsilon}$ where

$C \equiv \partial \mathcal{D}$ and $C_{s_0 \in \epsilon}$ is the portion of the contour lying in the interval $[s_0 - \epsilon, s_0 + \epsilon]$ and s_0 is the

location of the corner. Apply the above results separately to

contour C_1 and C_2 where $C_1 \equiv C - (s_0, s_0 + \epsilon)$ and $C_2 \equiv C - (s_0 - \epsilon, s_0)$. Since it is assumed that the curvature satisfies the Hölder condition on C_1 and C_2 (Hence the angle formed between the tangent to the contours and the Ox axis also satisfy the Hölder condition.) we have that $\Theta(s, \sigma)$ and $\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ satisfies the Hölder condition in S and σ for any $S \in C$ and $\sigma \in C - C_{s_0, \epsilon}$ as long as the arc length used is that of the portion of the contour which doesn't pass through s_0 . Since it is also true that when σ is restricted to lie on $C - C_{s_0, \epsilon}$ we have that $\Theta(s, \sigma)$ and $\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ are continuous in S even as S passes through the corner s_0 . We can conclude that $\Theta(s, \sigma)$ and $\frac{\partial \Theta}{\partial \sigma}(s, \sigma)$ satisfy the Hölder condition (with no restrictions on the choice of arc length) in $S \in C$ and $\sigma \in C - C_{s_0, \epsilon}$ for $\sigma \in C - C_{s_0, \epsilon}$.

2.2.

$$\left[R_1 \tan \frac{\bar{\beta}_1}{2} + \frac{2 R_1 \sin \frac{\theta_1}{2} \sin (\alpha + \frac{\theta_1}{2}) + R_2 [1 - \cos (\theta_2 + \bar{\beta}_2)]}{\sin (\alpha + \theta_1)} \right] \left[\frac{\sin \bar{\beta}_1}{\sin (\theta_1 + \alpha + \bar{\beta}_1)} \right] = \frac{2 R_2 \sin \frac{\bar{\beta}_2}{2} \sin (\theta_1 + \alpha + \theta_2 + \frac{\bar{\beta}_2}{2})}{\sin (\theta_1 + \alpha)}$$

The terms in the above equation are defined in figure A. Identical results are obtained with the geometry given in figure B by changing the signs of R_2 , θ_2 and $\bar{\beta}_2$. The derivation of the above equation follows:

$$\overline{O_1 S'_1} \perp \overline{S_2 S'_2} \quad ; \quad \overline{O_1 S'_1} \perp \overline{DB}$$

$$\overline{O_1 A} \perp \overline{IC} \quad ; \quad \overline{O_2 A} \perp \overline{KJ}$$

$$\overline{S'_1 G} \parallel \overline{O_2 A} \quad ; \quad \overline{S_2 G} \parallel \overline{KJ}$$

$$\overline{O_1K} \perp \overline{KJ}$$

$$\text{In } \Delta O_1S_1'B : \overline{S_1'B} = R_1 \tan \frac{\beta_1}{2} .$$

From the law of sines one obtains that in $\Delta AS_1'S_1''$

$$\frac{\overline{AS_1'}}{\overline{S_1''S_1'}} = \frac{\sin [\pi - (\alpha + \theta_1)]}{\sin (\alpha + \frac{\theta_1}{2})} \quad \text{where}$$

$$\overline{AS_1'} = 2R_1 \sin \frac{\theta_1}{2} , \text{ in } \Delta O_1AS_1' \quad ; \text{ hence}$$

$$\overline{S_1''S_1'} = \frac{2R_1 \sin \frac{\theta_1}{2} \sin (\alpha + \frac{\theta_1}{2})}{\sin (\alpha + \theta_1)} .$$

Now, since $\overline{S_1''B} = \overline{S_1''S_1'} + \overline{S_1'B}$ it follows that

$$\overline{S_1''B} = R_1 \tan \frac{\beta_1}{2} + \frac{2R_1 \sin \frac{\theta_1}{2} \sin (\alpha + \frac{\theta_1}{2})}{\sin (\alpha + \theta_1)} .$$

Furthermore, recalling that in $\Delta GS_1'D$,

$$\angle GS_1'D = \frac{\pi}{2} - [\pi - (\theta_1 + \alpha)] = \theta_1 + \alpha - \frac{\pi}{2}$$

$$\angle GDS_1' = \pi - (\theta_1 + \alpha) , \quad \text{and,}$$

$$\overline{GH} = R_2 - R_2 \cos (\beta_2 + \theta_2) \quad , \text{ it follows that}$$

$$\overline{S_1''D} = \frac{R_2 [1 - \cos(\bar{\beta}_2 + \theta_2)]}{\cos(\theta_1 + \alpha - \frac{\pi}{2})} = \frac{R_2 [1 - \cos(\bar{\beta}_2 + \theta_2)]}{\sin(\theta_1 + \alpha)}$$

Then

$$\overline{DB} = \overline{BS_1''} + \overline{S_1''D} \quad \text{becomes}$$

$$\begin{aligned} \overline{DB} = R_1 \tan \frac{\bar{\beta}_1}{2} + \frac{2 R_1 \sin \frac{\theta_1}{2} \sin(\alpha + \frac{\theta_1}{2})}{\sin(\theta_1 + \alpha)} + \\ + \frac{R_2 [1 - \cos(\theta_2 + \bar{\beta}_2)]}{\sin(\theta_1 + \alpha)}. \end{aligned}$$

One has that in $\triangle DBS_2$, $\angle DS_2B = \pi - (\theta_1 + \bar{\beta}_1 + \alpha)$, and according to the law of sines

$$\frac{\overline{BD}}{\sin[\pi - (\theta_1 + \bar{\beta}_1 + \alpha)]} = \frac{\overline{DS_2}}{\sin \bar{\beta}_1}, \quad \text{or}$$

$$\begin{aligned} \overline{DS_2} = \left[R_1 \tan \frac{\bar{\beta}_1}{2} + \frac{2 R_1 \sin \frac{\theta_1}{2} \sin(\alpha + \frac{\theta_1}{2}) + R_2 [1 - \cos(\theta_2 + \bar{\beta}_2)]}{\sin(\alpha + \theta_1)} \right] \frac{\sin \bar{\beta}_1}{\sin(\theta_1 + \alpha + \bar{\beta}_1)} \end{aligned}$$

(1)

Observing that in ΔDS_1S_2 :

$$\angle S_1DS_2 = \theta_1 + \alpha, \quad \text{and}$$

$$\angle DS_1S_2 = \angle ES_1S_2 - \angle ES_1D, \quad \text{where}$$

$$\angle ES_1S_2 = \frac{\pi}{2} - \frac{\bar{\beta}_2}{2}, \quad \text{and in } \Delta ES_1D$$

$$\begin{aligned} \angle ES_1D &= \pi - \left[\left(\frac{\pi}{2} - \theta_2 \right) + \pi - (\theta_1 + \alpha) \right] \\ &= \theta_1 + \alpha + \theta_2 - \frac{\pi}{2}, \quad \text{or} \end{aligned}$$

$$\angle DS_1S_2 = \pi - \left(\theta_1 + \alpha + \theta_2 + \frac{\bar{\beta}_2}{2} \right). \quad \text{Also,}$$

$\overline{S_1S_2} = 2R_2 \sin \frac{\bar{\beta}_2}{2}$, and from the law of sines one finally obtains that

$$\frac{\overline{S_1S_2}}{\overline{DS_2}} = \frac{\sin(\theta_1 + \alpha)}{\sin \left[\pi - \left(\theta_1 + \alpha + \theta_2 + \frac{\bar{\beta}_2}{2} \right) \right]}, \quad \text{or}$$

$$\overline{DS_2} = \frac{2R_2 \sin \frac{\bar{\beta}_2}{2} \sin \left(\theta_1 + \alpha + \theta_2 + \frac{\bar{\beta}_2}{2} \right)}{\sin(\theta_1 + \alpha)} \quad (2)$$

The desired equation is obtained by equating $\overline{DS_2}$ in eqn (1) and eqn (2).

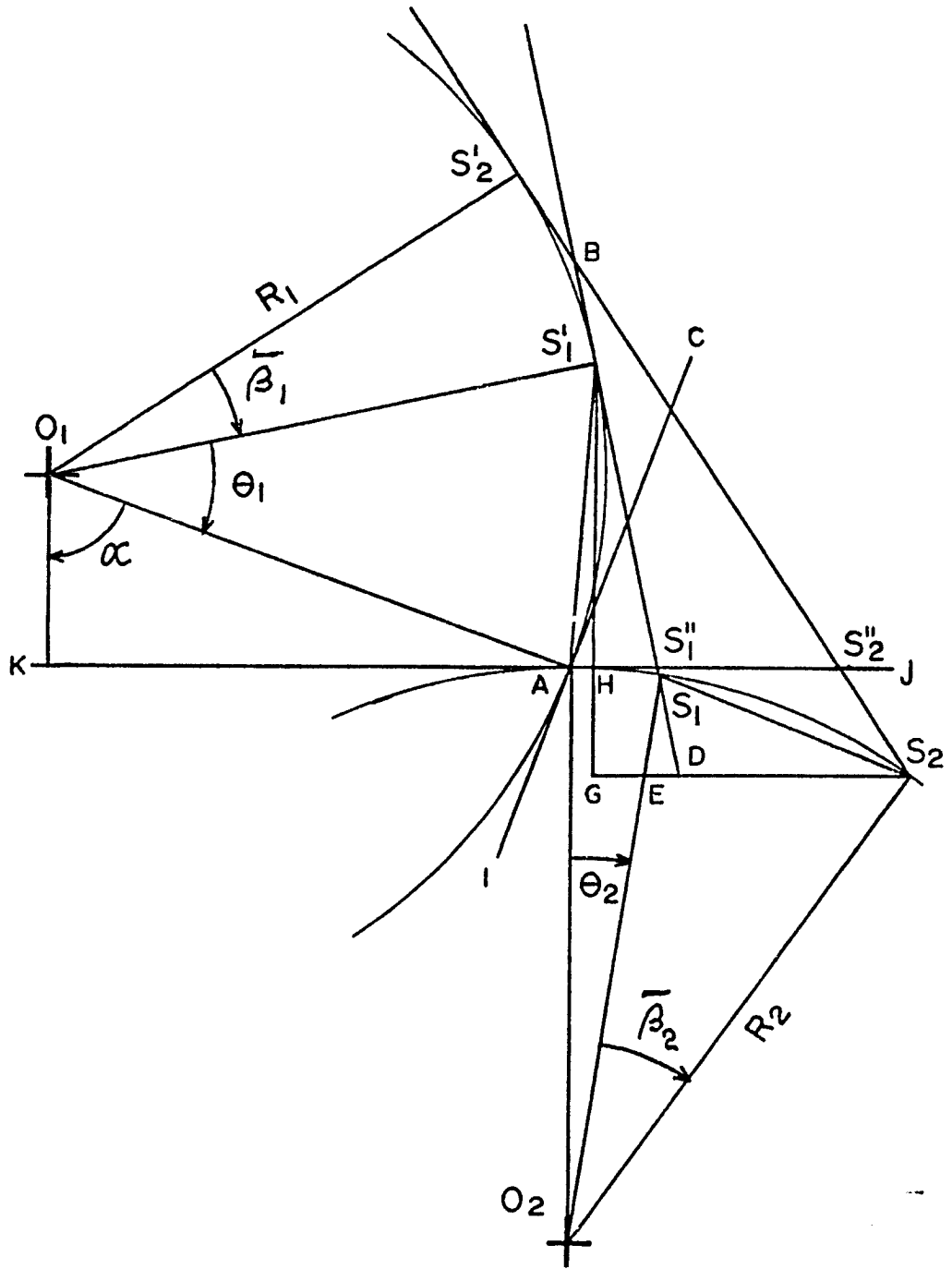


Figure A

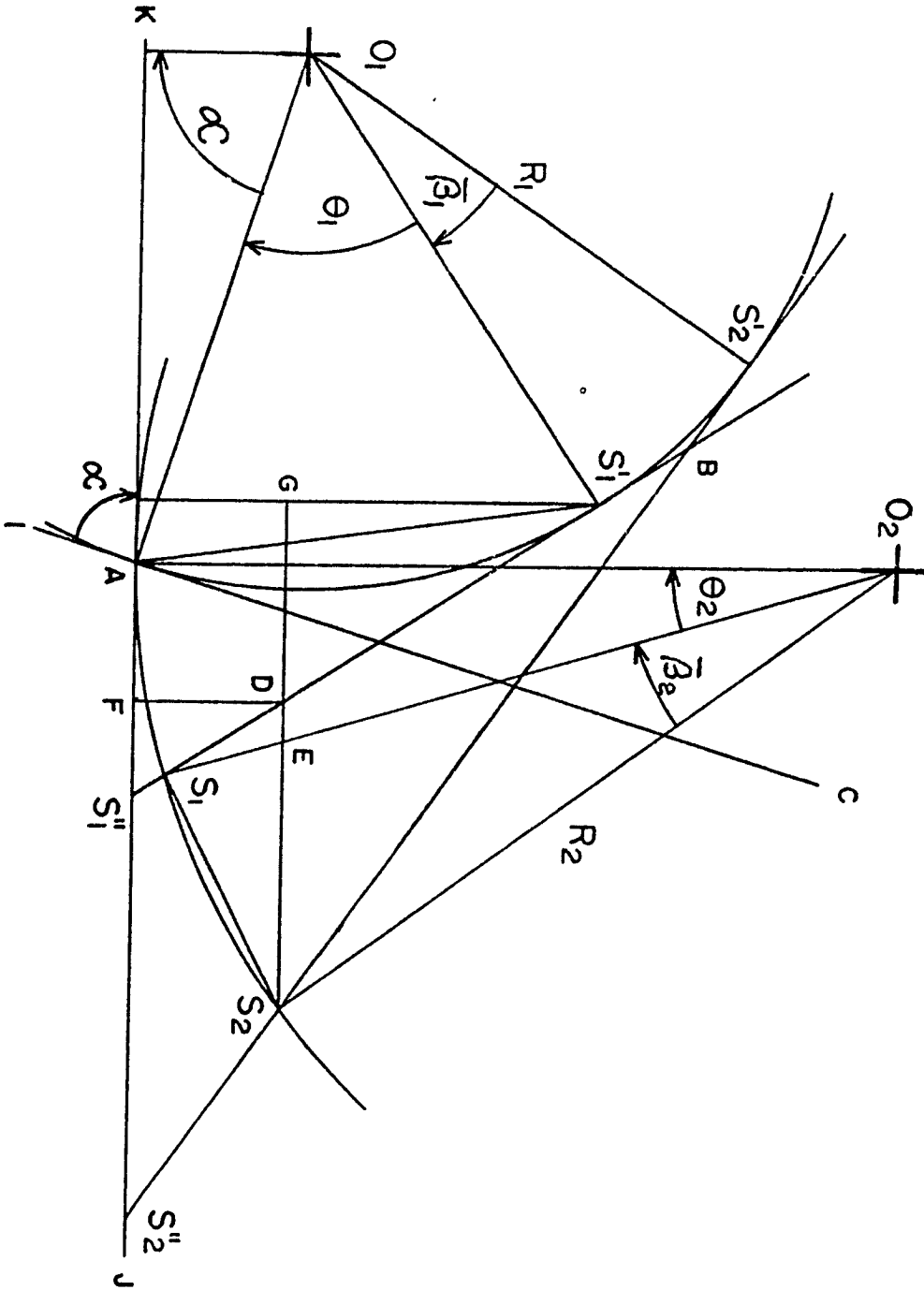


Figure B

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VITA

I was born on March 11, 1946.

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